Chern-Weil Theory

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Abstract

This paper gives an introduction to the theory of character classes and applications of the Chern class. We give some definitions from algebraic topology and differential geometry in case the reader is not familiar with these subjects. From these definitions we explore how we can use connections on vector bundles to construct classes of closed differential forms on a manifold that preserve the topological structure of the vector bundle. We will also provide a proof of the Chern-Gauss-Bonnet Theorem using Berezin integrals, and an analysis of the relationship between the Chern class and the Euler class using line bundles. After this classical introduction to characteristic forms, we then explore its applications to Čech cohomology and complex manifolds. We conclude the paper with a brief venture into index theory, where we prove the Chern-Gauss-Bonnet theorem and the Hirzebruch-Riemann-Roch theorem.

Contents

1 Survey of Algebraic Topology and Differential Geometry

The theory of characteristic classes is a beautiful intersection of algebraic topology and differential geometry. Since not all readers may be familiar with either of these subjects, we will explore the essential elements in this section. There will be many times where proofs of propositions go beyond the scope of this paper, so their proofs will be omitted, with references for the eager reader. I will assume that the reader has had some basic encounters with smooth manifolds and differential forms (particularly their integrability and Stoke's Theorem), although not necessarily in the full generality of this paper. Unless otherwise stated, M is assumed to be a smooth, connected, closed (compact and without boundary), orientable manifold of even dimension m.

1.1 Introduction to Differential Forms and de Rham Cohomology

We begin with a definition integral to our studies.

Definition 1. *Let E, B be topological spaces and $\pi : E \to B$ a continuous surjection. If $\pi^{-1}(x)$ is a finite-dimensional real (complex) vector space for each point $x \in B$ and if for all $x \in B$ there is a an open neighborhood U_x, a positive integer k_x, and a homeomorphism $\phi_x : U_x \times \mathbb{R}^{k_x} \to \pi^{-1}(U_x)$ ($\psi_x : U_x \times \mathbb{C}^{k_x} \to \pi^{-1}(U_x)$) such that*

1) $\pi \circ \phi_x(x, v) = x$ ($\pi \circ \psi_x(x, v) = x$)

2) $v \mapsto \phi_x(x, v)$ ($v \mapsto \psi_x(x, v)$) is a linear isomorphism between $\mathbb{R}^{k_x}$ ($\mathbb{C}^{k_x}$) and $\pi^{-1}(x)$

then we call (E, B, π) a real (complex) ***vector bundle****.*

We call E the ***total space****, B the* ***base space****, $\pi^{-1}(x)$ the* ***fiber at*** *x (which we will also often denote as E_x), and the collection of pairs (U_x, ϕ_x) ((U_x, ψ_x)) a* ***local trivialization****. Along connected components of B, $\pi^{-1}(x)$ is constant rank, which we will call the* ***rank of the vector bundle****. A vector bundle of rank 1 is called a* ***line bundle****. A vector bundle is often also denoted by the total space E or by the projection map π when the other elements of the collection are understood. To define a vector bundle, it is enough to define its fibers.*

A ***section*** *of $\pi : E \to B$ is a map $\sigma : B \to E$ such that $\pi \circ \sigma = \mathrm{id}_B$. We denote the space of smooth subsections of a bundle (E, B, π) as $\Gamma(B, E)$*[1]

There are many examples of vector bundles that we will explore in this paper.

Example 1. *For any vector space V, $B \times V$ is a vector bundle over B called the* ***trivial bundle of*** *V* ***and*** *B.*

Example 2. *For each $x \in M$, let T_xM be the tangent space of M at x. We can then construct the* ***tangent bundle of*** *M as the set $TM = \{(x, v) : x \in M, v \in T_xM\}$.*

Example 3. *For each $x \in M$, define $T_x^*M := (T_xM)^*$, the dual of T_xM. We call this space the* ***cotangent space of*** *M* ***at*** *x. We can then construct $T^*M := \{(x, \alpha) : x \in M, \alpha \in T_x^*M\}$, which is called the* ***cotangent bundle of*** *M. There is a pairing, denoted $(\cdot, \cdot)_T : TM \times T^*M \to \mathbb{R}$ where we apply the element of the cotangent space to the element of the tangent*

[1] Some authors choose to instead use the notation $\Gamma(E)$. This notation may cause issues if two vector bundles share the same total space, so we include the domain of the subsection to avoid any confusion.

space. Since M is assumed to be finite dimensional, we get that $(T^*M)^* = TM$, so we can also think of this pairing as applying the element of the tangent space to the element of the cotangent space.

*More generally, if E is a vector bundle over B, then there is a **dual vector bundle** E^* over B, where $E^*_x = (E_x)^*$.*

Example 4. *For any vector space V, the vector space of all linear maps $f : V \to V$ is called the **endomorphisms of** V and is denoted $\mathrm{End}(V)$. If (E, B, π) is a vector bundle, we can construct the vector bundle $\mathrm{End}(E)$, the **endomorphism bundle of** E such that $\mathrm{End}(E)_x := \mathrm{End}(E_x)$.*

*More generally, if (E, B, π) and (F, B, ϖ) are vector bundles over the same base space, we can construct the **hom-bundle of** E **and** F, defined as $\mathrm{Hom}(E, F)_x := \mathrm{Hom}(E_x, F_x)$, where $\mathrm{Hom}(V, W)$ is the set of all linear maps from a vector space V to a vector space W.*

Example 5. *Let (E, M, π) be a vector bundle and $f : N \to M$ a continuous map. We define the **pullback bundle** f^*E as the bundle over N such that $(f^*E)_x := E_{f(x)}$.*

Example 6. *Let (E, M, π) and (F, M, ϖ) be vector bundles. We define the **Whitney sum bundle** $E \oplus F$ as the bundle over N such that $(E \oplus F)_x := E_x \oplus F_x$ for all $x \in M$.*

*In an analogous way, we can define the **tensor bundle** $E \otimes F$ for any two vector bundles (E, B, π) and (F, B, ϖ). It is not too hard to see that if L is a trivial line bundle (with fibers over the same field as E) then $E \otimes L = E$. Similarly, there is a natural isomorphism between the bundles $E \otimes E^*$ and $\mathrm{End}(E)$.*

Definition 2. *Given a vector space V, we can construct the **tensor algebra of** V, denoted $T(V)$, as the algebra of tensor products of elements of V. We can further construct $\Lambda(V) := T(V)/\langle x \otimes x\rangle$ called the **exterior algebra of** V. To show the impact of this quotient on the tensor product[2], we typically use the symbol $\wedge$ instead of $\otimes$ to denote the tensor product. This tensor product is called the **wedge product**. We will also use the notation $\Lambda^k(V)$ as the subset of $\Lambda(V)$ consisting of the wedge products of k elements.*

There is a natural extension of the map $\pi : T^*M \to M$ to $\Lambda(T^*M) \to M$ such that $\Lambda(T^*M) \to M$ is a vector bundle. We then define $\Omega^k(M) := \Gamma\left(M, \Lambda^k(T^*M)\right)$ as the ***space of k-forms over*** M. From these, we can further construct $\Omega^*(M) := \bigoplus_{k=0}^{\infty} \Omega^k(M)$, the ***space of differential forms over*** M.

Define the operator $d : \Omega^*(M) \to \Omega^*(M)$ to be the unique linear map satisfying

1) df is the differential of f
2) $d(d\beta) = 0$
3) $d(\alpha \wedge \beta) = d\alpha \wedge \beta + (-1)^k \alpha \wedge d\beta$

where $f \in \Omega^0(M)$, $\alpha \in \Omega^k(M)$, and $\beta \in \Omega^*(M)$. For brevity, we often abbreviate $d(d\beta) = 0$ to $d^2\beta = 0$. This map is called the ***exterior derivative***. From this, we see that

$$d\Omega^k(M) \subseteq \ker d\big|_{\Omega^{k+1}(M)}$$

This property allows us to define the de Rham complex

$$0 \to \Omega^0(M) \xrightarrow{d} \Omega^1(M) \xrightarrow{d} \cdots \xrightarrow{d} \Omega^{\dim M}(M) \to 0 \tag{1}$$

We can then define the ***k-th de Rham cohomology***[3] as

$$H^k_{dR}(M; \mathbb{R}) := \frac{\ker d\big|_{\Omega^k(M)}}{d\Omega^{k-1}(M)} \tag{2}$$

[2]To see the impact of this quotient, we can compute $(x + y) \otimes (x + y)$ to show that $x \otimes y = -y \otimes x$ in the exterior algebra.

[3]Here we are assuming that our differential forms are $\mathbb{R}$-valued. If they are $\mathbb{C}$-valued, we could analogously construct $H^k_{dR}(M; \mathbb{C})$. All of our theorems for $H^k_{dR}(M; \mathbb{R})$ hold for $H^k_{dR}(M; \mathbb{C})$.

Further, we construct the ***total de Rham cohomology*** as

$$H_{dR}(M;\mathbb{R}) := \bigoplus_{k=0}^{\dim M} H_{dR}^k(M;\mathbb{R}) \tag{3}$$

We will make use of the following propositions from homological algebra. A proof of the following can all be found in [5].

Proposition 1. *If $f : M \to N$ is a continuous map, then there is a homomorphism $f^* : H^\bullet(N;\mathbb{R}) \to H^\bullet(M;\mathbb{R})$. The symbol $\bullet$ is meant to show that f^* restricts to maps on fixed degrees.*

Proposition 2. *Let U, V be open sets such that $U \cup V = M$, then there is the short exact sequence*

$$0 \to H_{dR}^\bullet(M;\mathbb{R}) \to H_{dR}^\bullet(U;\mathbb{R}) \oplus H_{dR}^\bullet(V;\mathbb{R}) \to H_{dR}^\bullet(U\cap V;\mathbb{R}) \to 0 \tag{4}$$

*called the **Mayer-Vietoris sequence**. It induces a long exact sequence*

$$\begin{aligned} 0 \to H_{dR}^0(M;\mathbb{R}) \to H_{dR}^0(U;\mathbb{R}) \oplus H_{dR}^0(V;\mathbb{R}) \to H_{dR}^0(U\cap V;\mathbb{R}) \to \\ H_{dR}^1(M;\mathbb{R}) \to \cdots \to H_{dR}^m(U\cap V;\mathbb{R}) \to 0 \end{aligned} \tag{5}$$

also called the Mayer-Vietoris sequence.

Proposition 3 (Poincaré Duality)**.** *Let M be a manifold of dimension n. The inner product on $\Omega^k(M) \times \Omega^{n-l}(M)$ given by $\langle \alpha, \beta \rangle = \int_M \alpha \wedge \beta$ gives a duality between $H_{dR}^k(M;\mathbb{R})$ and $H_{dR}^{n-k}(M;\mathbb{R})$.*

Definition 3. *Let $f : N \to M$ be a map between two manifolds. We define $\Omega^k(f) := \Omega^k(M) \oplus \Omega^{k-1}(N)$ and $d(\alpha, \beta) := (d\alpha, f^*\alpha - d\beta)$. We then have the short exact sequence*

$$0 \to \Omega^{k-1}(N) \overset{\beta \mapsto (0,\beta)}{\to} \Omega^k(f) \overset{(\alpha,\beta)\mapsto\alpha}{\to} \Omega^k(M) \to 0 \tag{6}$$

This induces a long exact sequence on on cohomology

$$0 \cdots \to H_{dR}^{k-1}(N;\mathbb{R}) \to H_{dR}^k(f;\mathbb{R}) \to H_{dR}^k(M;\mathbb{R}) \overset{f^*}{\to} H_{dR}^k(N;\mathbb{R}) \to \cdots \tag{7}$$

We see that cohomology classes in $H_{dR}^k(f;\mathbb{R})$ represent a closed form on M which pull back to an exact form on N.

*If N is taken to be a submanifold of M and $\iota : N \to M$ be the inclusion, we define the **relative cohomology** $H_{dR}^k(M, N;\mathbb{R}) := H_{dR}^k(\iota;\mathbb{R})$.*

Definition 4. *We say that a differential form α is **closed** if $d\alpha = 0$ and we say that it is **exact** if there exists a differential form β such that $\alpha = d\beta$.*

With Definition 4, we can redefine the de Rham cohomology of M as

$$H_{dR}^k(M;\mathbb{R}) := \frac{\text{closed k-forms}}{\text{exact k-forms}} \tag{8}$$

This quotient allows us to define equivalence classes, called ***cohomology classes***, of closed forms. We say that two closed forms α, β define the same cohomology class if they differ by an exact form. We denote the cohomology class of a closed form α as $[\alpha]$. We see that cohomology classes satisfy the following properties

$$[a\alpha] = a[\alpha] \tag{9}$$
$$[\alpha + \beta] = [\alpha] + [\beta] \tag{10}$$

where a is a constant function. Further

$$(\alpha + d\beta) \wedge (\gamma + d\delta) = \alpha \wedge \gamma + d\left(\beta \wedge \gamma + \beta \wedge d\delta + (-1)^{\deg \alpha} \alpha \wedge \delta\right)$$

so the cohomology class of $[\alpha \wedge \gamma]$ depends only on $[\alpha]$ and $[\gamma]$. We write this fact as $[\alpha \wedge \beta] = [\alpha] \wedge [\beta]$. Combined with our results above, we see that $H^*_{dR}(M;\mathbb{R})$ carries a natural ring structure. Importantly, for $0 \leq k \leq \dim(M)$, $\dim H^k_{dR}(M;\mathbb{R})$ is finite dimensional[4] and the de Rham cohomology agrees with the singular cohomology.

One important topological property that comes from $H_{dR}(M;\mathbb{R})$ is the Euler characteristic.

Definition 5. *The* ***Euler characteristic of*** M*,* $\chi(M)$*, is defined as*

$$\chi(M) := \sum_{i=0}^{n} (-1)^i \dim H^i_{dR}(M;\mathbb{R}) \tag{11}$$

Proposition 4. *If* M *is a closed surface of genus* g*, then* $\chi(M) = 2 - 2g$*.*

The Euler characteristic, by construction, is a purely topological property. As a consequence, if two topological spaces X, Y are homeomorphic, $\chi(X) = \chi(Y)$, but the converse is not true. This quantity will be very important throughout the paper.

1.2 Vector Bundles and Differential Forms

Since the total space E of a real vector bundle (E, M, π) is a topological space, it has de Rham cohomology groups. However, we will need to create a slightly more restrictive cohomology to gain useful information about of the vector bundles. For a function $f : X \to \mathbb{R}$, where X is some topological space, we define the ***support of*** f as $\operatorname{supp} f := \{p \in X : f(p) \neq 0\}$. Let $\Omega^k_{cv}(E) = \left\{\alpha \in \Omega^k(E) : \alpha \text{ has compact support along each fiber}\right\}$. From this, we can construct the ***compact vertical de Rham cohomology*** $H^*_{cv}(E;\mathbb{R})$ in a similar manner to above.

Proposition 5 (Leray-Hirsch Theorem)**.** *Let* E *be a vector bundle over* M*. If there are global cohomology classes* $[e_1], \ldots, [e_k]$ *on* E *which when restricted to each fiber freely generate the cohomology of the fiber, then* $H^*_{cv}(E;\mathbb{R})$ *is a free module over* $H^*_{dR}(M;\mathbb{C})$ *with basis* $\{[e_1], \ldots, [e_k]\}$*. In other words*

$$H^*_{cv}(E;\mathbb{R}) = H^*_{dR}(M;\mathbb{R}) \otimes \mathbb{R}\{[e_1], \ldots, [e_k]\} \tag{12}$$

Proposition 6. *If* (E, M, π) *is a rank* n *real oriented vector bundle, then there is a map* $\pi_* : \Omega^\bullet_{cv}(E) \to \Omega^{\bullet - n}(M)$ *such that* $\pi_*((\pi^*\tau) \wedge \omega) = \tau \wedge \pi_*\omega$ *for* $\tau \in \Omega^*(M)$ *and* $\omega \in \Omega^*_{cv}(E)$*. We call this map* ***integration along the fiber****.*

Proof. We will assume that the vector bundle is trivial (the general case follows from locally trivializing the vector bundle and piecing the local trivializations together). Let $t_1, \ldots, t_n$ be coordinates on the fiber $\mathbb{R}^n$. Then a form on E is a real linear combination of forms that do not contain a factor of the n-form $dt_1 \wedge \cdots \wedge t_n$ and those that do. Let β be a form on M, $\pi^*\beta$ its pullback to E, and $f : E \to \mathbb{R}$ a function with compact support along each fixed fiber.

For $r < n$, we define π_* as follows:

$$\pi_*(\pi^*\beta f(x, t_1, \ldots, t_n)\, dt_{i_1} \wedge \cdots \wedge dt_{i_r}) := 0 \tag{13}$$

[4]It is not immediate why $H^k_{dR}(M;\mathbb{R})$ should be finite-dimensional. If we think of the k-th de Rham cohomology by the definition in Equation 8, the space of closed forms is by no means finite-dimensional, and neither is the space of exact forms. The fact that $H^k_{dR}(M;\mathbb{R})$ is finite dimensional is entirely due to our assumption that M be compact. For a proof, refer to Proposition 1.5.3.1 in [5].

Alternatively,

$$\pi_* \left(\pi^* \beta f(x, t_1, \dots, t_n)\, dt_1 \wedge \cdots \wedge dt_n\right) := \beta \int_{\mathbb{R}^n} f\, dt_1 \wedge \cdots \wedge dt_n \tag{14}$$

If $\omega = \pi^* \beta f(x, t_1, \dots, t_n)\, dt_{i_1} \wedge \cdots \wedge dt_{i_r}$, then

$$\begin{aligned}
\pi_* \left((\pi^* \tau) \wedge \omega\right) &= \pi_* \left((\pi^* \tau) \wedge \pi^* \beta f(x, t_1, \dots, t_n)\, dt_{i_1} \wedge \cdots \wedge dt_{i_r}\right) \\
&= \pi_* \left((\pi^* \tau \beta) f(x, t_1, \dots, t_n)\, dt_{i_1} \wedge \cdots \wedge dt_{i_r}\right) \\
&= 0
\end{aligned}$$

which is trivially equal to $\tau \wedge \pi_* \omega$. If ω is of the second type, then

$$\begin{aligned}
\pi_* \left((\pi^* \tau) \wedge \omega\right) &= \pi_* \left((\pi^* \tau \beta) f(x, t_1, \dots, t_n)\, dt_1 \wedge \cdots \wedge dt_n\right) \\
&= \tau \beta \int_{\mathbb{R}^n} f(x, t_1, \dots, t_n)\, dt_1 \wedge \cdots \wedge dt_n \\
&= \tau \wedge \pi_* \omega
\end{aligned}$$

□

This map π_* induces a map on the cohomologies $\pi_* : H^{\bullet}_{cv}(E; \mathbb{R}) \to H^{\bullet - n}_{dR}(M; \mathbb{R})$. In fact, as a map on cohomologies, this map is an isomorphism (Theorem 6.17 in [5]). We call the inverse map the ***Thom isomorphism*** T.

The Thom isomorphism gives a map $H^0_{dR}(M; \mathbb{R}) \to H^n_{cv}(E; \mathbb{R})$. As such, the image of 1 under T determines a cohomology class $\Phi(E)$, called the ***Thom class of*** E. By the proposition above, if $\alpha \in H^*_{dR}(M; \mathbb{R})$

$$\begin{aligned}
\pi_* \left(\pi^* \alpha \wedge \Phi\right) &= \alpha \wedge \pi_* \Phi \\
&= \alpha \wedge 1 \\
&= \alpha
\end{aligned}$$

So, we have an explicit formula for T given by $T(\alpha) = \pi^* \alpha \wedge \Phi$.

Proposition 7. *If E, F are real oriented vector bundles over M, let $p_E : E \oplus F \to E$ and $p_F = E \oplus F \to F$ be the projection maps. Then $\Phi(E \oplus F) = p_E^* \Phi(E) \wedge p_F^* \Phi(F)$*

Proof. See Prop 6.19 in [5]. □

The Thom form gives rise to another important form in Chern-Weil theory. If $\sigma \in \Gamma(M, E)$ is the zero subsection of E (the subsection that sends $x \in M$ to $0 \in \pi^{-1}(x)$), then we define the ***euler class*** of E to be $e(E) := \sigma^* \Phi(E)$.

Proposition 8. *Let $\pi : E \to M$ be an oriented vector bundle over M of rank n and E^0 the compliment in E of the zero section. Then there is a globally defined form $\psi(E)$ on E^0 such that $d\psi(E) = -\pi^* e(E)$. Further, $(\Phi(E), \psi(E))$ represents a class in $H^n_{cv}\left(E, E^0\right)$.*

Proof. See Chapter 1.6 in [5] □

Moving beyond just real and complex-valued forms, if $E \xrightarrow{\pi} M$ is a complex vector bundle, we construct $\Omega^*(M; E) := \Gamma(M, \Lambda(T^* M) \otimes E)$, the ***space of differential forms on*** M ***with values in*** E. It is often useful to think of an element of $\Omega^*(M; E)$ as an element of the form $\Omega^*(M) \Gamma(M, E)$.

Definition 6. *Given a vector bundle E, a **connection on** E, is a linear map $\nabla^E : \Gamma(M, E) \to \Omega^1(M; E)$ such that for any $f \in C^\infty(M)$ and $X \in \Gamma(M, E)$,*

$$\nabla^E(fX) = (df)\, X + f \nabla^E X \tag{15}$$

We can also extend ∇^E to a map from $\Omega^\bullet(M;E) \to \Omega^{\bullet+1}(M;E)$ such that $\alpha X \mapsto (d\alpha)X + (-1)^k \alpha \wedge \nabla^E X$, where $\alpha \in \Omega^k(M)$ and $X \in \Gamma(M,E)$. In this manner, a connection can be thought of as an extension of the exterior derivative to include the coefficient E. Unfortunately, the important property $d^2 = 0$ may not necessarily be true. We call the obstruction to this property the curvature of a connection.

Definition 7. *The* ***curvature*** *of a connection is defined as* $R^E := \nabla^E \circ \nabla^E : \Gamma(M,E) \to \Omega^2(M;E)$.

Proposition 9. *The curvature* R^E *is* C^∞*-linear.*

Proof. If $f \in C^\infty(M)$ and $X \in \Gamma(M,E)$, then

$$\begin{aligned} R^E(fX) &= \nabla^E\left((df)X + f\nabla^E X\right) \\ &= d^2 f X + (-1)^{\deg df}(df) \wedge \nabla^E X + (df) \wedge \nabla^E X + fR^E X \\ &= fR^E X \end{aligned}$$

□

If we think more about what R^E is doing, it takes a map from $M \to E$ to a 2-form with values in E. By the linearity of the connection, this is the same as a 2-form with values in $\mathrm{End}(E)$, so $R^E \in \Omega^2(M;\mathrm{End}(E))$. Importantly, if we think of the extension of ∇^E to a map $\Omega^\bullet(M;E) \to \Omega^{\bullet+1}(M;E)$, then $\left(R^E\right)^k = R^E \circ \cdots \circ R^E \in \Omega^{2k}(M;\mathrm{End}(E))$.

Returning to our connection, let $Y \in \Gamma(M,TM)$. If $(\cdot,\cdot)_T$ is the paring of the tangent and cotangent bundles, we can define the ***covarient derivative of*** ∇^E ***along*** Y, denoted $\nabla^E_Y : \Gamma(M,E) \to \Gamma(M,E)$, as the map $\nabla^E_Y(X) = \left(Y, \nabla^E(X)\right)_T$. The Leibniz rule of the connection then becomes

$$\nabla^E_Y(fX) = (Yf)X + f\nabla^E_Y X \tag{16}$$

where $X \in \Gamma(M,E)$ and $f \in C^\infty(M)$.

Definition 8. *A* ***Riemannian metric on*** TM g *is an inner product on each* T_xM *that varies smoothly along* TM. *If* $e_1, \ldots, e_n$ *is an oriented orthonormal basis of* TM, *we can construct a dual basis* $e^1, \ldots, e^n$ *on* T^*M *by* $e^i = ge_i$.

More generally, one can define a ***bundle metric*** $\langle\cdot,\cdot\rangle_E$, or e for short, as an object that is a metric on each fiber and varies smoothly as we move along the total space[5]. If $e_1, \ldots, e_n$ is an oriented orthonormal basis of E, we can construct a dual basis on E^* in an analogous way. We say that a connection ∇^E on a bundle E is ***metric*** or ***metric compatible*** if for $X \in \Gamma(M,TM)$, $X\langle u,v\rangle_E = \langle\nabla^E_X u, v\rangle_E + \langle u, \nabla^E_X v\rangle_E$ for any two $u, v \in \Gamma(M,E)$.

If we define $\nabla^E e : \Gamma(M,E) \times \Gamma(M,E) \times \Gamma(M,TM) \to \mathbb{F}$ (where $\mathbb{F}$ is the underlying field of the vector bundle) as the map $(u,v,X) \mapsto X\langle u,v\rangle_E - \langle\nabla^E_X u, v\rangle_E - \langle u, \nabla^E_X v\rangle_E$, then we see that ∇^E being metric is equivalent to $\nabla^E e \equiv 0$.

Given a Riemannian metric g on TM, there is a canonical connection on TM called the ***Levi-Civita*** connection, denoted by ∇^{TM}. An important property of the Levi-Civita connection is that it is compatible with g.

Proposition 10. *If* $\langle\cdot,\cdot\rangle_E$ *is a bundle metric and* ∇^E *is compatible, then* R^E *is skew-symmetric.*

A proof of the above proposition can be found in Appendix A.

If M is two-dimensional and $E = TM$, then the curvature of the Levi-Civita metric R^{TM} is of the form $R^{TM} = \begin{pmatrix} 0 & -K \\ K & 0 \end{pmatrix} dA$, where dA is the area element and K is called the ***Gaussian curvature***[6].

[5] The existence of such an object is not obvious. If the base space is paracompact (which for our purposes, it is), then we can pullback the Euclidean structure from $\mathbb{R}^n$ onto the local trivialization, and then glue them together by partitions of unity.

[6] For justification for this statement, see Appendix A

1.3 The Hodge Star Operator on Riemannian Manifolds

Definition 9. *The* ***Hodge star operator*** *with respect to a Riemannian metric g on TM is a map $\star : \Lambda(T^*M) \to \Lambda(T^*M)$ given by $e^1 \wedge \cdots \wedge e^k \mapsto e^{k+1} \wedge \cdots \wedge e^n$. In a more technical sense, let $I = \{i_1, \ldots, i_k\}$ be an ordered set of k distinct elements of $\{1, \ldots, n\}$ and $J = \{j_1, \ldots, j_{n-k}\}$ be its ordered compliment. Then, $\star e^{i_1} \wedge \cdots \wedge e^{i_k} = \epsilon_{I,J} e^{j_1} \wedge \cdots \wedge e^{j_{n-k}}$, where $\epsilon_{I,J}$ is given by $\left(e^{i_1} \wedge \cdots \wedge e^{i_k}\right) \wedge \left(e^{j_1} \wedge \cdots \wedge e^{j_{n-k}}\right) = \epsilon_{I,J} e^1 \wedge \cdots \wedge e^n$.*

The Hodge star operator extends naturally to a map on $\Omega^*(M) \to \Omega^*(M)$.

Proposition 11. *The Hodge star operator has the following properties:*

1) For any k, $\star\star = (-1)^{nk+k} : \Omega^k(M) \to \Omega^k(M)$

2) For any k and $\alpha, \beta \in \Omega^k(M)$, $\alpha \wedge \star\beta = \beta \wedge \star\alpha$

3) $\alpha \wedge \star\alpha = 0$ if and only if $\alpha = 0$.

Proof. For property 1),

$$\begin{aligned}\star\star\, dx^{i_1} \wedge \cdots \wedge dx^{i_k} &= \epsilon_{I,J} \star dx^{j_1} \wedge \cdots \wedge dx^{j_{n-k}} \\ &= \epsilon_{I,J}\epsilon_{J,I} dx^{i_1} \wedge \cdots \wedge dx^{i_k} \\ &= (-1)^{k(n-k)}\, dx^{i_1} \wedge \cdots \wedge dx^{i_k} \\ &= (-1)^{kn+k}\, dx^{i_1} \wedge \cdots \wedge dx^{i_k}\end{aligned}$$

For property 2), by linearity, we may assume that $\alpha = dx^{i_1} \wedge \cdots \wedge dx^{i_k}$ and $\beta = dx^{j_1} \wedge \cdots \wedge dx^{j_k}$. We then see that $\alpha \wedge \star\beta$ and $\beta \wedge \star\beta$ are both n-forms. By the structure of the exterior algebra, if $\{i_1, \ldots, i_k\} \neq \{j_1, \ldots, j_k\}$ (now thought of as unordered sets), then $\alpha \wedge \star\beta = \beta \wedge \star\alpha = 0$. Otherwise, if I' and J' are the ordered compliments of $I = \{i_1, \ldots, i_k\}$, $J = \{j_1, \ldots, j_k\}$, then

$$\begin{aligned}\alpha \wedge \star\beta &= \epsilon_{J,J'} \alpha \wedge \left(dx^{j'_1} \wedge \cdots \wedge dx^{j'_{n-k}}\right) \\ &= \epsilon_{J,J'}\epsilon_{I,J'} dx^1 \wedge \cdots \wedge dx^n\end{aligned}$$

If we denote by $\epsilon^{I,J}$ the constant such that $\alpha = \epsilon^{I,J}\beta$, then

$$\begin{aligned}\alpha \wedge \star\beta - \beta \wedge \star\alpha &= \epsilon_{J,J'}\epsilon_{I,J'} dx^1 \wedge \cdots \wedge dx^n - \epsilon_{J,I'}\epsilon_{I,I'} dx^1 \wedge \cdots \wedge dx^n \\ &= \left(\epsilon_{J,J'}\epsilon_{I,J'} - \epsilon_{J,I'}\epsilon_{I,I'}\right) dx^1 \wedge \cdots \wedge dx^n \\ &= \left(\left(\epsilon^{J',I'}\right)^2 \epsilon_{J,I'}\epsilon_{I,I'} - \epsilon_{J,I'}\epsilon_{I,I'}\right) dx^1 \wedge \cdots \wedge dx^n \\ &= \left(\epsilon_{J,I'}\epsilon_{I,I'} - \epsilon_{J,I'}\epsilon_{I,I'}\right) dx^1 \wedge \cdots \wedge dx^n \\ &= 0\end{aligned}$$

For Property 3), we can assume that $\alpha = f(x)\, dx^{i_1} \wedge \cdots \wedge dx^{i_k}$. Then

$$\begin{aligned}\alpha \wedge \star\alpha &= \epsilon_{I,J} f(x) \left(dx^{i_1} \wedge \cdots \wedge dx^{i_k}\right) \wedge \left(dx^{j_i} \wedge \cdots \wedge dx^{j_{n-k}}\right) \\ &= \epsilon_{I,J}^2 f(x) \\ &= f(x)\end{aligned}$$

As such, $\alpha \wedge \star\alpha = 0$ if and only if $f(x) \equiv 0$, implying that $\alpha = 0$. □

Corollary 1. *There is an inner product $\langle \cdot, \cdot \rangle_\Omega$ on $\Omega^\bullet(M)$ given by*

$$\langle \alpha, \beta \rangle_\Omega := \int_M \alpha \wedge \star\beta \tag{17}$$

Definition 10. *Let $d^* : \Omega^*(M) \to \Omega^*(M)$ be the operator defined by*

$$d^*(\alpha) := (-1)^{n \deg \alpha + n + 1} \star d \star \alpha \tag{18}$$

As such, for any k, $d^* : \Omega^k(M) \to \Omega^{k-1}(M)$.

Proposition 12. *d^* is the formal adjoint of d.*

Proof. Let $\alpha \in \Omega^k(M)$ and $\beta \in \Omega^{k+1}(M)$, then

$$\begin{aligned}
\langle d\alpha, \beta \rangle_\Omega - \langle \alpha, d^*\beta \rangle_\Omega &= \int_M d\alpha \wedge \star\beta - \alpha \wedge \star d^*\beta \\
&= \int_M d\alpha \wedge \star\beta + (-1)^k \alpha \wedge d \star \beta \\
&= \int_M d(\alpha \wedge \star\beta) \\
&= \int_{\partial M} a \wedge \star\beta \\
&= 0
\end{aligned}$$

With the last line being true because ∂M is empty. □

Definition 11. *The* ***de Rham-Hodge operator*** *D associated with Riemannian metric g is defined as*

$$D := d + d^* : \Omega^*(M) \to \Omega^*(M) \tag{19}$$

If we define $\Omega^{\text{even}}(M) := \bigoplus_{i \text{ even}} \Omega^i(M)$ and Ω^{odd} similarly, then we see that D restricts to maps $D_{\text{even}} : \Omega^{\text{even}} \to \Omega^{\text{odd}}$ and $D_{\text{odd}} : \Omega^{\text{odd}} \to \Omega^{\text{even}}$. Clearly, they are formal adjoints.

We then define the ***Laplacian*** of D to be

$$\begin{aligned}
\Delta &:= D^2 && (20)\\
&= dd^* + d^*d && (21)
\end{aligned}$$

With this tool, we state

Theorem 1. *$\Omega^*(M)$ decomposes as $\Omega^*(M) = \ker \Delta \oplus \operatorname{Im} \Delta$.*

If we restrict to certain degrees, then we could restate this decomposition as $\Omega^k(M) = \ker \Delta_k \oplus \operatorname{Im} d_{k-1} \oplus \operatorname{Im} d^*_{k+1}$. We call elememts of $\ker \Delta$ ***harmonic***.

Corollary 2. *For any k, $\ker \Delta_k \cong H^k_{dR}(M; \mathbb{R})$.*

Proof. Let $\alpha \in \ker \Delta_k$. Then

$$\begin{aligned}
\langle d\alpha, d\alpha \rangle_\Omega + \langle d^*\alpha, d^*\alpha \rangle_\Omega &= \langle \Delta\alpha, \alpha \rangle_\Omega \\
&= 0
\end{aligned}$$

This implies that $\langle d\alpha, d\alpha \rangle_\Omega = 0$ which implies that $d\alpha = 0$, making it closed. If $\beta \in [\alpha]$, then α and β differ by an exact form $d\gamma$. But our decomposition of $\Omega(M)$ tells us that since $\beta = \alpha - d\gamma \in \ker \Delta_k$, $d\gamma = 0$, so $\alpha = \beta$. So, each element of $\ker \Delta_k$ corresponds to a unique element of $\Omega^k(M)$.

Alternatively, if ω is a closed k-form, our decomposition says that since $d\omega = 0$, $\omega = \omega' + d\omega''$ for $\omega' \in \ker \Delta_k$ and $\omega'' \in \operatorname{Im} d_{k-1}$. But this assigns to each closed k-form an element of $\ker \Delta_k$ □

Our proof of the above corollary gives the following remark.

Remark 1. $\ker \Delta = \ker (D)$.

Corollary 3. *If M is a Riemannian manifold, then*

$$\chi (M) = \ker (D_{even}) - \ker (D_{odd}) \tag{22}$$

Proof. By Corollary 2 and our remark above,

$$\begin{aligned}\chi (M) &= \sum_{i=1}^{\dim M} (-1)^i \dim H^i_{dR} (M; \mathbb{R}) \\ &= \sum_{i=1}^{\dim M} (-1)^i \dim \ker \Delta_i \\ &= \sum_{i=1}^{\dim M} (-1)^i \dim \ker D\big|_{\Omega^i(M)} \\ &= \dim \ker D_{\text{even}} - \dim \ker D_{\text{odd}}\end{aligned}$$

□

2 Characteristic Forms, Chern Form, and Beyond

Let (E, M, π) be a vector bundle and $A \in \Gamma (M, \operatorname{End} (E))$. Then the fiberwise trace of A creates a smooth function on M denoted $\operatorname{tr} [A]$. This function induces the map $\operatorname{tr} : \Omega^* (M; \operatorname{End} (E)) \to \Omega^* (M)$ such that $\alpha A \mapsto \alpha \operatorname{tr} [A]$, for $\alpha \in \Omega^* (M)$ and $A \in \Gamma (M, \operatorname{End} (E))$. We still call this function the ***trace***.

We define the ***Lie bracket*** on $\operatorname{End} (E)$ as $[X, Y] = XY - YX$. If $\alpha, \beta \in \Omega^* (M)$ and $A, B \in \Gamma (M, \operatorname{End} (E))$, we can extend the Lie bracket to $\Omega^* (M; \operatorname{End} (E))$ via

$$[\alpha A, \beta B] = (\alpha A) (\beta B) - (-1)^{(\deg \alpha)(\deg \beta)} (\beta B) (\alpha A) \tag{23}$$

Proposition 13. *For $A, B \in \Omega^* (M; \operatorname{End} (E))$, $\operatorname{tr} [[A, B]] = 0$.*

Proof. Let $A = \alpha a$ and $B = \beta b$, where $\alpha, \beta \in^* \Omega (M)$ and $a, b \in \Gamma (M, \operatorname{End} (E))$. Then,

$$\begin{aligned}\operatorname{tr} [[A, B]] &= \operatorname{tr} \left[(\alpha a) (\beta b) - (-1)^{(\deg \alpha)(\deg \beta)} (\beta b) (\alpha a)\right] \\ &= \alpha \wedge \beta \operatorname{tr} [ab] - (-1)^{(\deg \alpha)(\deg \beta)} \beta \wedge \alpha \operatorname{tr} [ba] \\ &= \left(\left(1 - (-1)^{(\deg \alpha)(\deg \beta)} (-1)^{(\deg \alpha)(\deg \beta)}\right) \alpha \wedge \beta\right) \operatorname{tr} [ab] \\ &= 0\end{aligned}$$

□

Lemma 1. *If ∇^E is a connection on E, then for any $A \in \Omega^* (M; \operatorname{End} (E))$,*

$$d \operatorname{tr} [A] = \operatorname{tr} \left[\left[\nabla^E, A\right]\right] \tag{24}$$

Proof. We first need to show that this claim is independent of our choice of connection. Let $\tilde{\nabla}^E$ be another connection on E. We see that for $X \in \Gamma (M, E)$ and $f \in C^\infty$

$$\begin{aligned}\left(\nabla^E - \tilde{\nabla}^E\right) fX &= (df) X + f \nabla^E X - (df) X - f \tilde{\nabla}^E X \\ &= f \left(\nabla^E - \tilde{\nabla}^E\right) X\end{aligned}$$

So, we can think of $\nabla^E - \tilde{\nabla}^E$ as an element of $\Omega^1(M; \mathrm{End}(E))$. We then have that

$$\begin{aligned} 0 &= \mathrm{tr}\left[\left[\nabla^E - \tilde{\nabla}^E, A\right]\right] \\ &= \mathrm{tr}\left[\left[\nabla^E, A\right]\right] - \mathrm{tr}\left[\left[\tilde{\nabla}^E, A\right]\right] \end{aligned}$$

So, the right-hand side of Equation 24 does not depend on our choice of connection. The second half of the proof follows from the locality of the right-hand side of Equation 24 and trivializations of the vector bundle E. □

Let $f(x) = \sum_{k=0}^{\infty} a_k x^k$ be a power series. Since R^E is even-degree, as is each of its compositions, $f\left(R^E\right) \in \Omega^*(M; \mathrm{End}(E))$ is well defined. We see then that $\mathrm{tr}\left[f\left(R^E\right)\right] \in \Omega^*(M)$, from which, we have the following theorem.

Theorem 2 (Chern-Weil Theorem)**.** *The differential form* $\mathrm{tr}\left[f\left(R^E\right)\right]$ *is closed and if* $\tilde{\nabla}^E$ *is another connection on* E *with curvature* $\tilde{R}^E$*, then* $\mathrm{tr}\left[f\left(R^E\right)\right]$ *and* $\mathrm{tr}\left[f\left(\tilde{R}^E\right)\right]$ *are in the same cohomology class.*

Proof. From Lemma 1, we have that

$$\begin{aligned} d\,\mathrm{tr}\left[f\left(R^E\right)\right] &= \mathrm{tr}\left[\left[\nabla^E, f\left(R^E\right)\right]\right] \\ &= \mathrm{tr}\left[\sum_{k=0}^{\infty} a_k \left[\nabla^E, \left(R^E\right)^k\right]\right] \\ &= \mathrm{tr}\left[\sum_{k=0}^{\infty} a_k \underbrace{\left[\nabla^E, \left(\nabla^E\right)^{2k}\right]}_{=0}\right] \\ &= 0 \end{aligned}$$

This proves the first claim. For the second claim, let $t \in [0,1]$ and define $\nabla_t^E := t\nabla^E + (1-t)\tilde{\nabla}^E$. Since convex combinations of connections are connections, ∇_t^E is a connection on E. Define R_t^E as the curvature of ∇_t^E. We see that

$$\begin{aligned} \frac{d}{dt}\mathrm{tr}\left[f\left(R_t^E\right)\right] &= \mathrm{tr}\left[\frac{dR_t^E}{dt} f'\left(R_t^E\right)\right] \\ &= \mathrm{tr}\left[\frac{d\left(\nabla_t^E\right)^2}{dt} f'\left(R_t^E\right)\right] \\ &= \mathrm{tr}\left[\left[\nabla_t^E, \frac{d\nabla_t^E}{dt}\right] f'\left(R_t^E\right)\right] \\ &= \mathrm{tr}\left[\left[\nabla_t^E, \frac{d\nabla_t^E}{dt} f'\left(R_t^E\right)\right]\right] \\ &= d\,\mathrm{tr}\left[\frac{d\nabla_t^E}{dt} f'\left(R_t^E\right)\right] \\ \implies \mathrm{tr}\left[f\left(\tilde{R}^E\right)\right] - \mathrm{tr}\left[f\left(R^E\right)\right] &= d\int_0^1 \mathrm{tr}\left[\frac{d\nabla_t^E}{dt} f'\left(R_t^E\right)\right] dt \end{aligned}$$

□

Corollary 4. *For any power series* f*,* $\mathrm{tr}\left[f\left(\frac{\sqrt{-1}}{2\pi}R^E\right)\right]$ *is a closed differential form with cohomology class* $\left[\mathrm{tr}\left[f\left(\frac{\sqrt{-1}}{2\pi}R^E\right)\right]\right]$*. Further, this class does not depend on our choice of* *connection on E. We call this differential form the* ***characteristic form of*** E ***associated with*** ∇^E ***and*** f *(denoted as* $f\left(E, \nabla^E\right)$*) and the cohomology class is called the* ***characteristic class of*** E ***associated with*** f *(denoted* $f(E)$*).*

Let $E_1, \ldots, E_k$ be vector bundles over M with choices of connections $\nabla^{E_1}, \ldots, \nabla^{E_k}$ and $f_1, \ldots, f_k$ be power series. Since each characteristic form is a closed differential form, their sum or product is also a closed differential form, and we will also call this sum or product a characteristic form. So, we construct the characteristic form $f_1\left(E_1, \nabla^{E_1}\right) \cdots f_k\left(E_k, \nabla^{E_k}\right)$. Let $\left\{f_1\left(E_1, \nabla^{E_1}\right) \cdots f_k\left(E_k, \nabla^{E_k}\right)\right\}^{\max}$ denote its component in $\Omega^{\dim M}(M)$.

Lemma 2. *The number defined by*

$$\int_M f_1\left(E_1, \nabla^{E_1}\right) \cdots f_k\left(E_k, \nabla^{E_k}\right) = \int_M \left\{f_1\left(E_1, \nabla^{E_1}\right) \cdots f_k\left(E_k, \nabla^{E_k}\right)\right\}^{max} \tag{25}$$

is independent of the choice of connection.

Proof. Without loss of generality, assume that $\tilde{\nabla}^{E_1}$ is another connection on E_1. Then $f_1\left(E_1, \nabla^{E_1}\right) - f_1\left(E_1, \tilde{\nabla}^{E_1}\right) = d\alpha$ for some differential form α on M. If we denote by χ and $\tilde{\chi}$ $\int_M f_1\left(E_1, \nabla^{E_1}\right) \cdots f_k\left(E_k, \nabla^{E_k}\right)$ and $\int_M f_1\left(E_1, \tilde{\nabla}^{E_1}\right) \cdots f_k\left(E_k, \nabla^{E_k}\right)$ respectively, then

$$\begin{aligned}
\chi - \tilde{\chi} &= \int_M d\alpha f_2\left(E_2, \nabla^{E_2}\right) \cdots f_k\left(E_k, \nabla^{E_k}\right) \\
&= \int_M d\left(\alpha f_2\left(E_2, \nabla^{E_2}\right) \cdots f_k\left(E_k, \nabla^{E_k}\right)\right) \\
&= \int_{\partial M} \alpha f_2\left(E_2, \nabla^{E_2}\right) \cdots f_k\left(E_k, \nabla^{E_k}\right) \\
&= 0
\end{aligned}$$

Since $f_2\left(E_2, \nabla^{E_2}\right) \cdots f_k\left(E_k, \nabla^{E_k}\right)$ is a closed form, its exterior derivative is zero, so we achieve the second equality by the Leibniz rule of the exterior derivative. We then employ Stoke's theorem and since ∂M is empty, the intergal vanishes. ☐

We call the number given by Equation 25 the ***characteristic number of the characteristic class*** $f_1(E_1) \cdots f_k(E_k)$. We denote it by $\langle f_1(E_1) \cdots f_k(E_k), [M]\rangle$.

Definition 12. *Let E be a complex vector bundle over M, ∇^E a connection, and R^E its curvature tensor. The* ***total chern form with respect to*** ∇^E*, denoted* $c\left(E, \nabla^E\right)$*, is*

$$c\left(E, \nabla^E\right) := \det\left(I + \frac{\sqrt{-1}}{2\pi} R^E\right) \tag{26}$$

where I is the identity endomorphism of E. Since $\det = \exp \circ \operatorname{tr} \circ \log$*, and* $\exp$ *and* $\log$ *have well-defined power series, we see that* $c\left(E, \nabla^E\right)$ *is a characteristic form. We then denote its characteristic class as* $c(E)$ *and call it the* ***total chern class of*** E*.*

The power series expansion of exp and the definition of $c\left(E, \nabla^E\right)$ makes it clear that $c\left(E, \nabla^E\right)$ has a decomposition $c\left(E, \nabla^E\right) = 1 + c_1\left(E, \nabla^E\right) + \cdots + c_k\left(E, \nabla^E\right) + \cdots$ with each $c_i\left(E, \nabla^E\right) \in \Omega^{2i}(M)$. We call these the ***$i$-th chern forms with respect to*** ∇^E and their cohomology classes $c_i(E)$.

Proposition 14. *Let E, F be complex vector bundles over M. Then*

1) *If $k > \operatorname{rank} E$, then $c_k(E) = 0$. We call $c_{\operatorname{rank} E}(E)$ the* ***top chern class of*** E*.*

2) *$c(E \oplus F) = c(E)\, c(F)$.*

Proof. See section 4 in [5]. ☐

Another important form is the ***Chern character form*** (not to be confused with the total Chern form), which is defined as

$$\mathrm{ch}\left(E,\nabla^{E}\right) := \mathrm{tr}\left[\exp\left(\frac{\sqrt{-1}}{2\pi}R^{E}\right)\right] \tag{27}$$

We denote its cohomology class as $\mathrm{ch}\,(E)$, called the ***Chern character***.

Proposition 15. *If E, F are complex vector bundles, then*

1) $\mathrm{ch}\,(E \oplus F) = \mathrm{ch}\,(E) \oplus \mathrm{ch}\,(F)$

2) $\mathrm{ch}\,(E \otimes F) = \mathrm{ch}\,(E)\,\mathrm{ch}\,(F)$

Proof. See section 1.5 in [3]. Please note that [3] uses a slightly different definition of the Chern character form, but the results are still the same. ☐

Stepping away from the name Chern, we take a look at another form, the ***Todd form***, defined as

$$\mathrm{Td}\left(E,\nabla^{E}\right) = \det\left(\frac{\frac{\sqrt{-1}}{2\pi}R^{E}}{I - \exp\left(-\frac{\sqrt{-1}}{2\pi}R^{E}\right)}\right) \tag{28}$$

For the same reasons as above, the Todd form is a characteristic form and its cohomology class is denoted by $\mathrm{Td}\,(E)$. Similar to chern classes, $\mathrm{Td}\,(E \oplus F) = \mathrm{Td}\,(E)\,\mathrm{Td}\,(F)$ [3]. As we will explore in the remainder of this paper, these three classes are very connected. One such connection is below.

Proposition 16. *Let E be a complex vector bundle of rank n over M. Then $\mathrm{ch}\,(E)$ and $\mathrm{Td}\,(E)$ can be written as polynomials in the $c_i\,(E)$. Explicitly,*

$$\mathrm{ch}\,(E) = n + c_1 + \frac{c_1^2 - 2c_2}{2} + \frac{c_1^3 - 3c_1c_2 - 3c_3}{6} + \cdots \tag{29}$$

$$\mathrm{Td}\,(E) = 1 + \frac{c_1}{2} + \frac{c_1^2 + c_2}{12} \tag{30}$$

where the evaluation at E has been suppressed to save space.

3 The Chern-Gauss-Bonnet Theorem

Let E be an oriented, Euclidean vector space of dimension n. We can think of this as a vector bundle over a point. Let $\mathbf{x} = \left(x^1, \ldots, x^n\right)$ be an oriented Euclidean coordinate system on E and set

$$U\,(\mathbf{x}) = \left(\frac{1}{2\pi}\right)^{n/2} e^{-|\mathbf{x}|^2/2} dx^1 \wedge \cdots \wedge dx^n \tag{31}$$

It is a known result that

$$\int_E U = 1 \tag{32}$$

Definition 13. *We define the **Berezin integral**[7] $\int^B : \Lambda(E) \to \mathbb{R}$ as the map*

$$\int^B : \alpha \mapsto \langle \alpha, dx^1 \wedge \cdots \wedge dx^n \rangle \tag{33}$$

in other words, if $e_1, \ldots, e_n$ is an oriented orthonormal basis of E and $ae_1 \wedge \cdots \wedge e_n$ is the component of $\alpha \in \Lambda(E)$ of degree n, then $\int^B \alpha = a$.

We can lift $\Lambda(E)$ to a vector bundle over E. We can then extend the Berezin integral to a map from $\Omega^*(E; \Lambda(E)) \to \Omega(E)$. If $\alpha \in \Omega^*(E)$ and $\beta \in \Gamma(E, \Lambda(E))$, then

$$\int^B : \alpha \wedge \beta \mapsto \alpha \int^B \beta \tag{34}$$

Consider the identity map on E as an element of $\Omega^0(E; E)$. Then its exterior differential $d\mathbf{x} \in \Omega^1(E; E)$. We then have the following proposition.

Proposition 17. *The function $U(\mathbf{x})$ as defined above is equivalent to*

$$U(\mathbf{x}) = (-1)^{\frac{n(n+1)}{2}} \left(\frac{1}{2\pi}\right)^{n/2} \int^B \exp\left(-\frac{|\mathbf{x}|^2}{2} - d\mathbf{x}\right) \tag{35}$$

Proof. We first note that $\exp\left(-\frac{|\mathbf{x}|^2}{2} - d\mathbf{x}\right) = \exp\left(-\frac{|\mathbf{x}|^2}{2}\right) \wedge \exp(-d\mathbf{x})$, with $\exp\left(-\frac{|\mathbf{x}|^2}{2}\right) \in \Omega^*(E)$ and $\exp(-d\mathbf{x}) \in \Omega^1(E; \Lambda(E))$. Additionally, we can write $\mathbf{x} = x^1 e_1 + \cdots x^n e_n$. As such,

$$\begin{aligned}
(-1)^{\frac{n(n+1)}{2}} \left(\frac{1}{2\pi}\right)^{n/2} \int^B \exp(-d\mathbf{x}) &= (-1)^{\frac{n(n+1)}{2}} \left(\frac{1}{2\pi}\right)^{n/2} \int^B \exp(-d\mathbf{x}) \\
&= (-1)^{\frac{n(n+1)}{2}} \left(\frac{1}{2\pi}\right)^{n/2} \int^B \prod_{k=1}^{n} \left(1 - dx^k \wedge e_k\right) \\
&= (-1)^{\frac{n^2+3n}{2}} \left(\frac{1}{2\pi}\right)^{n/2} \int^B \left(dx^1 \wedge e_1\right) \wedge \cdots \wedge \left(dx^n \wedge e_n\right) \\
&= \left(\frac{1}{2\pi}\right)^{n/2} \int^B \left(dx^1 \wedge \cdots dx^n\right) \wedge \left(e_1 \wedge \cdots \wedge e_n\right) \\
&= \left(\frac{1}{2\pi}\right)^{n/2} dx^1 \wedge \cdots dx^n
\end{aligned}$$

We then have

$$\begin{aligned}
(-1)^{\frac{n(n+1)}{2}} \left(\frac{1}{2\pi}\right)^{n/2} \int^B \exp\left(-\frac{|\mathbf{x}|^2}{2} - d\mathbf{x}\right) &= (-1)^{\frac{n(n+1)}{2}} \left(\frac{1}{2\pi}\right)^{n/2} \exp\left(-\frac{|\mathbf{x}|^2}{2}\right) \int^B \exp(-d\mathbf{x}) \\
&= \left(\frac{1}{2\pi}\right)^{n/2} \exp\left(-\frac{|\mathbf{x}|^2}{2}\right) dx^1 \wedge \cdots \wedge dx^n \\
&= U(\mathbf{x})
\end{aligned}$$

$\square$

In one last extension, let $\pi : E \to M$ be an oriented Euclidean vector bundle of rank n. Then we can fiberwise extend the Berezin integral to define a map

$$\int^B : \Omega^*(M; \Lambda(E)) \mapsto \Omega^*(M) \tag{36}$$

[7] This map is not an integral in the usual sense. In some ways it acts more like a derivative. For more of an understanding of this object, we refer the reader to [15].

If ∇^E is a connection on E that preserves the metric on E, then it extends naturally to an action ∇ on $\Omega^*(M;\Lambda(E))$.

Proposition 18. *For $\alpha \in \Omega^*(M;\Lambda(E))$,*

$$d\int^B \alpha = \int^B \nabla\alpha \tag{37}$$

Proof. Without loss of generality, we may assume that $\alpha = \beta e_1 \wedge \cdots e_n$ for some $\beta \in \Omega^*(M)$. Then,

$$\begin{aligned}
\int^B \nabla\alpha &= \int^B (d\beta)\, e_1 \wedge \cdots \wedge e_n + (-1)^{\deg\beta} \wedge \nabla(e_1 \wedge \cdots \wedge e_n) \\
&= \int^B (d\beta)\, e_1 \wedge \cdots \wedge e_n \\
&= d\beta \\
&= d\int^B \alpha
\end{aligned}$$

□

Let π^*E be the pullback bundle over M. If ∇^E preserves the Euclidean metric on E, then it extended to a connection on π^*E that also preserves the metric and thus is a derivation on $\Omega^*(E;\Lambda(\pi^*E))$.

Definition 14. *If $\pi : E \to M$ is a Euclidean vector bundle and $\sigma \in \Gamma(E,\Lambda(E))$ is a smooth section, then we define the **interior multiplication with respect to** σ as the formal adjoint of the wedge product, that is $\langle \iota_\sigma x, y\rangle = \langle x, y \wedge \sigma\rangle$. It extends naturally to a derivation on $\Omega^*(E;\Lambda(\pi^*E))$*

Since interior multiplication decreases the degrees in $\Lambda(\pi^*E)$, we can rewrite Proposition 18 as

$$d\int^B \alpha = \int^B (\nabla + \iota_\sigma)\,\alpha \tag{38}$$

for any $\alpha \in \Omega^*(E;\Lambda(\pi^*E))$ and $\sigma \in \Gamma(E,\pi^*E)$.

Remark 2. *We can identify $\mathfrak{so}(E)$, the subset of $\operatorname{End}(E)$ consisting of skew-adjoint elements with $\Lambda^2(E)$ by the map*

$$A \in \mathfrak{so}(E) \mapsto \sum_{i<j} \langle Ae_i, e_j\rangle e_i \wedge e_j \tag{39}$$

Corollary 5. *If ∇^E is a connection that agrees with the metric on E, we can think of $R^E \in \Omega^2(M;\operatorname{End}(E))$ now as an element of $\Omega^2\left(M;\Lambda^2(E)\right)$.*

Lemma 3. *Let $\mathbf{x} \in \Omega^0(E;\pi^*E)$, such that $|\mathbf{x}|^2 \in \Omega^0(E)$ and $\nabla\mathbf{x} \in \Omega^1\left(E;\Lambda^1(\pi^*E)\right)$, and let π^*R^E be the pullback of R^E. Define*

$$\mathcal{A} := \frac{|\mathbf{x}|^2}{2} + \nabla\mathbf{x} - \pi^*R^E \in \Omega^*(E;\Lambda(\pi^*E))$$

Then $(\nabla + \iota_{\mathbf{x}})\,\mathcal{A} = 0$

Proof.

$$
\begin{aligned}
(\nabla + \iota_{\mathbf{x}})\,\mathcal{A} &= \nabla \frac{|\mathbf{x}|^2}{2} + \nabla\nabla\mathbf{x} - \nabla\pi^* R^E + \iota_{\mathbf{x}} \frac{|\mathbf{x}|^2}{2} + \iota_{\mathbf{x}}\nabla\mathbf{x} - \iota_{\mathbf{x}}\pi^* R^E \\
&= -\iota_{\mathbf{x}}\nabla\mathbf{x} + \pi^* R^E\mathbf{x} - 0 + 0 + \iota_{\mathbf{x}}\nabla\mathbf{x} - \iota_{\mathbf{x}}\pi^* R^E \\
&= \pi^* R^E\mathbf{x} - \iota_{\mathbf{x}}\pi^* R^E \\
&= \iota_{\mathbf{x}}\pi^* R^E - \iota_{\mathbf{x}}\pi^* R^E \\
&= 0
\end{aligned}
$$

□

With this, we now generalize our function $U(\mathbf{x})$ to be

$$
U(\mathbf{x}) = (-1)^{n(n+1)/2} \left(\frac{1}{2\pi}\right)^{n/2} \int^{B} \exp(-\mathcal{A}) \tag{40}
$$

By Equation 38 and Lemma 3, U is a closed n-form on E and defines a cohomology class. If we also restrict to fibers, then by Equation 32, we see that the fiber-wise integration of U is 1. These facts combined tell us that U is a Thom form of E[8]. Further, the cohomology class of U is the Thom class of E.

Let us modify our definition of $\mathcal{A}$ slightly to

$$
\mathcal{A}_t := \frac{t^2 |\mathbf{x}|^2}{2} + t\nabla\mathbf{x} - \pi^* R^E \tag{41}
$$

and let U_t be the Thom form constructed from $\mathcal{A}_t$.

Proposition 19.

$$
\frac{dU_t}{dt} = -(-1)^{n(n+1)/2} \left(\frac{1}{2\pi}\right)^{n/2} d\int^{B} \left(\mathbf{x}e^{-\mathcal{A}_t}\right) \tag{42}
$$

Proof. We see that

$$
\begin{aligned}
\frac{d\mathcal{A}_t}{dt} &= t\,|\mathbf{x}|^2 + \nabla\mathbf{x} \\
&= (\nabla + t\iota_{\mathbf{x}})\,\mathbf{x}
\end{aligned}
$$

Since we are now working with $\mathcal{A}_t$, Lemma 3 takes the form $(\nabla + t\iota_{\mathbf{x}})\,\mathcal{A}_t = 0$. As such,

$$
\begin{aligned}
\frac{d}{dt}\exp(-\mathcal{A}_t) &= -\frac{d\mathcal{A}_t}{dt}\exp(-\mathcal{A}_t) \\
&= -(\nabla + t\iota_{\mathbf{x}})\,(\mathbf{x}\exp(-\mathcal{A}_t))
\end{aligned}
$$

Which implies

$$
\begin{aligned}
\frac{dU_t}{dt} &= dU(\mathbf{x}) \\
&= (-1)^{n(n+1)/2} \left(\frac{1}{2\pi}\right)^{n/2} \int^{B} \frac{d}{dt}\exp(-\mathcal{A}_t) \\
&= -(-1)^{n(n+1)/2} \left(\frac{1}{2\pi}\right)^{n/2} \int^{B} (\nabla + t\iota_{\mathbf{x}})\,(\mathbf{x}\exp(-\mathcal{A}_t)) \\
&= -(-1)^{n(n+1)/2} \left(\frac{1}{2\pi}\right)^{n/2} d\int^{B} \left(\mathbf{x}e^{-\mathcal{A}_t}\right)
\end{aligned}
$$

□

[8] An astute reader may notice that our definition of a Thom form required compact support along the fibers, whereas U has decaying support on the fibers. We refer such readers to [12] for how to alieviate their concern.

If we take the pullback of U by a section $\sigma \in \Gamma(M, E)$,

$$\sigma^* U = (-1)^{n(n+1)/2} \left(\frac{1}{2\pi}\right)^{n/2} \int^B \exp\left(-\left(\frac{|\sigma|^2}{2} + \nabla\sigma - R^E\right)\right) \tag{43}$$

Importantly, if we take the zero section, we get the Euler form of E

$$0^* U = (-1)^{n(n+1)/2} \left(\frac{1}{2\pi}\right)^{n/2} \int^B \exp\left(R^E\right) \tag{44}$$

We call $\int^B \exp\left(R^E\right)$ the ***Pfaffian*** of R^E and denote it as $\operatorname{Pf}\left(R^E\right)$. Therefore, the Euler class of E is $e(E) = (-1)^{n(n+1)/2} \left(\frac{1}{2\pi}\right)^{n/2} \operatorname{Pf}\left(R^E\right)$. Importantly, if $E = TM$, the Euler class is independent of choice of metric on TM.

We can now state the goal of this paper: the Chern-Gauss-Bonnet Theorem.

Theorem 3 (Chern-Gauss-Bonnet)**.** *If M is a smooth closed manifold of even dimension, then the characteristic number of $e(TM)$ is $\chi(M)$.*

Proof. Let $V \in \Gamma(M, TM)$ such that the zero set of V, which we will denote $\operatorname{zero}(V)$ is discrete and non-degenerate—for any $x \in \operatorname{zero}(V)$, there is an oriented coordinate system $\mathbf{y} = \left(y^1, \ldots, y^n\right)$ on a small neighborhood U_x of x such that near x, $V(\mathbf{y}) = \mathbf{y}A + O\left(|\mathbf{y}|^2\right)$ for some invertible $n \times n$ matrix A (that does not depend on $\mathbf{y}$). The existence of such a subsection is a well-known result of differential topology.

From Prop 19, for any $t > 0$

$$\begin{aligned} \frac{dU_t}{dt} &= -\left(\frac{-1}{2\pi}\right)^{n/2} d \int^B \left(\mathbf{x} \exp\left(-\mathcal{A}_t\right)\right) \\ \implies V^* U_t - 0^* U_t &= -\left(\frac{-1}{2\pi}\right)^{n/2} d \int_0^1 \int^B \left(\mathbf{x} \exp\left(-\mathcal{A}_t\right)\right) \end{aligned}$$

which implies that the cohomology class of $V^* U$ is the Euler class. From this, we see that

$$\begin{aligned} \left(\frac{-1}{2\pi}\right)^{n/2} \int_M \operatorname{Pf}\left(R^{TM}\right) &= \int_M V^* U_t \\ &= \left(\frac{-1}{2\pi}\right)^{n/2} \int_M \int^B \exp\left(-\left(\frac{t^2 |V|^2}{2} + t\nabla^{TM} V - R^{TM}\right)\right) \end{aligned}$$

For any $x \in \operatorname{zero}(V)$, we can choose local coordinates such that $V(\mathbf{y}) = \mathbf{y}A_x$. Further, since the Euler class is metric-independent, we will assume that in each U_x for $x \in \operatorname{zero}(V)$, the metric g^{TM} is of the form $\left(dy^1\right)^2 + \cdots + \left(dy^n\right)^2$. As such,

$$\begin{aligned} \left(\frac{-1}{2\pi}\right)^{n/2} \int_M \operatorname{Pf}\left(R^{TM}\right) &= \left(\frac{-1}{2\pi}\right)^{n/2} \int_M \int^B \exp\left(-\left(\frac{t^2 |V|^2}{2} + t\nabla^{TM} V - R^{TM}\right)\right) \\ &= \left(\frac{-1}{2\pi}\right)^{n/2} \int_{M \setminus \bigcup_{x \in \operatorname{zero}(V)} U_x} \int^B \exp\left(-\left(\frac{t^2 |V|^2}{2} + t\nabla^{TM} V - R^{TM}\right)\right) \\ &\ldots + \sum_{x \in \operatorname{zero}(V)} \left(\frac{-1}{2\pi}\right)^{n/2} \int_{U_x} \int^B \exp\left(-\left(\frac{t^2 |V|^2}{2} + t\nabla^{TM} V - R^{TM}\right)\right) \end{aligned}$$

Since $|V|$ is bounded below by a positive number on $M \setminus \bigcup_{x \in \text{zero}(V)} U_x$, we see that as $t \to \infty$, the integral over $M \setminus \bigcup_{x \in \text{zero}(V)} U_x$ goes to zero. So, as $t \to \infty$

$$\begin{aligned}
\left(\frac{-1}{2\pi}\right)^{n/2} \int_M \mathrm{Pf}\left(R^{TM}\right) &= \sum_{x \in \text{zero}(V)} \left(\frac{-1}{2\pi}\right)^{n/2} \int_{U_x} \int^B \exp\left(-\left(\frac{t^2 |V|^2}{2} + t\nabla^{TM} V - R^{TM}\right)\right) \\
&= \sum_{x \in \text{zero}(V)} \left(\frac{-1}{2\pi}\right)^{n/2} \int_{U_x} \int^B \exp\left(-\left(\frac{t^2 |V|^2}{2} + tdV\right)\right) \\
&= \sum_{x \in \text{zero}(V)} \left(\frac{-1}{2\pi}\right)^{n/2} \int_{U_x} \int^B \exp\left(-\left(\frac{t^2 |\mathbf{y}A_x|^2}{2} + td\left(\mathbf{y}A_x\right)\right)\right) \\
&= \sum_{x \in \text{zero}(V)} \left(\frac{1}{2\pi}\right)^{n/2} t^n \det A_x \int_{U_x} \exp\left(-\left(\frac{t^2 |\mathbf{y}A_x|^2}{2}\right)\right) dy^1 \wedge \cdots \wedge dy^n \\
&= \sum_{x \in \text{zero}(V)} \left(\frac{1}{2\pi}\right)^{n/2} t^n \,\mathrm{sign}\left(\det A_x\right) \left|\det A_x\right| \int_{U_x} \exp\left(-\left(\frac{t^2 |\mathbf{y}A_x|^2}{2}\right)\right) dy^1 \wedge \cdots \wedge dy^n \\
&= \sum_{x \in \text{zero}(V)} \mathrm{sign}\left(\det A_x\right)
\end{aligned}$$

We now cite an important theorem in differential topology.

Lemma 4 (Poincaré-Hopf index formula)**.** *If $V \in \Gamma(M, TM)$ such that the zero set of V, is discrete and non-degenerate, then*

$$\chi(M) = \sum_{x \in \text{zero}(V)} \mathrm{sign}\left(\det A_x\right) \tag{45}$$

From which, we find that

$$\begin{aligned}
\langle e(TM), [M] \rangle &= \int_M e(TM) \\
&= \left(\frac{-1}{2\pi}\right)^{n/2} \int_M \mathrm{Pf}\left(R^{TM}\right) \\
&= \sum_{x \in \text{zero}(V)} \mathrm{sign}\left(\det A_x\right) \\
&= \chi(M)
\end{aligned}$$

□

We offer and alternative proof of Theorem 3 in Appendix A.

4 Chern Class and Line Bundles

As the proofs of many of the propositions and theorems in this subsection rely on principals of algebraic topology beyond the scope of this paper, we refer the eager reader to section 4 of [5] for proofs of the statements below.

4.1 Breaking Up the Chern Class

Proposition 20. *Let $(E, M, \pi), (F, N, \varpi)$ be vector bundles and $f : M \to N$ a continuous map. Then,*

$$c(E) = f^*\left(c(F)\right) \tag{46}$$

where f^* is the map given by Proposition 1.

An important consequence of Proposition 20 is that if E, F are isomorphic as vector bundles, then they have the same chern class. Another important consequence is given below.

Corollary 6. *Let (E, M, π) be trivial. Then $c_i(E) = 0$ for all i.*

Proof. Let (E, p, ϖ) be a trivial vector bundle over a point space p (note that both vector bundles have the same total space). There is a continuous map $f : M \to p$. If we let $f^*(E)$ be the pullback vector bundle over M, then it is clear that $f^*(E)$ is equivalent to (E, M, π). We then have $c_i(E, M, \pi) = f^* c_i(E, p, \varpi)$. Since p is a point space and $c_i \in H^{2i}_{dR}(p)$, $c_i(E, p, \varpi) = 0$, so $0 = f^* c_i(E, p, \varpi) = c_i(E, M, \pi)$. $\square$

Recall that a real (complex) line bundle L is a rank-1 real (complex) vector bundle.

Theorem 4. *If a real (complex) line bundle (L, M, π) has a non-vanishing section σ, then the line bundle is trivial, or in other words, $L \cong M \times \mathbb{R}$ ($L \cong M \times \mathbb{C}$).*

Proof. Let us assume that (L, M, π) is a complex line bundle (the argument for real line bundle will be identical as we will use none of the additional properties of $\mathbb{C}$). Let $\rho : M \times \mathbb{C} \to L$ be the map sending $(x, a) \mapsto a\sigma(x)$. I claim that ρ is continuous. Since L is a complex vector bundle, it has a local trivialization (U, ψ). For a neighborhood U in our trivialization, we see that $\psi \circ \sigma\big|_U : U \to U \times \mathbb{C}$ is continuous, i.e. for all $x \in U$, $x \mapsto \left(x, f\left(\sigma\big|_U(x)\right)\right)$ for some continuous function $f : \pi^{-1}(U) \to \mathbb{C}$. As such, $\psi \circ \rho(x, a) = \left(x, f\left(a\sigma\big|_U(x)\right)\right)$ which is continuous. Since ψ is a homeomorphism, ρ must be continuous.

Since ρ is continuous, consider the map $\rho_x : \mathbb{C} \to L_x$ for fixed $x \in M$ given by $a \mapsto \rho(x, a) = a\sigma(x)$. Since σ is nowhere vanishing, we see that ρ_x is injective. As an injective linear map between 1-dimensional vector spaces, it must be an isomorphism.

Let $v \in L$. Then we can continuously assign it to the element $\left(\pi(v), \rho^{-1}_{\pi(v)}(v)\right) \in M \times \mathbb{C}$. Similarly, if we have an element $(x, a) \in M \times \mathbb{C}$, we can continuously assign it to the element $\rho(x, a)$. Importantly, these maps are inverses, making ρ an isomorphism. $\square$

Proposition 21. *For two complex line bundles K, L over M, $c_1(K \otimes L) = c_1(K) + c_1(L)$.*

Corollary 7. *Let L be a line bundle over M. Then if L^* denotes the dual bundle of L (which is a still a line bundle) $L \otimes L^* = \mathrm{End}(L)$ has a non-vanishing subsection (namely, the identity), so it is trivial. As such, $0 = c_1(L \otimes L^*) = c_1(L) + c_1(L^*)$, implying that $c_1(L^*) = -c_1(L)$.*

We now state the following theorems.

Theorem 5. *Let (E, π, M) be a complex vector bundle of rank n. Then there exists a space $F(E)$ and a map $\sigma : F(E) \to E$ such that $\sigma^* H_{dR}(E) \to H_{dR}(F(E))$ is injective and the pullback bundle $\sigma^* E \to F(E)$ breaks up into a direct sum of complex line bundles $L_1, \ldots, L_n$, i.e. $\sigma^* E = L_1 \oplus \cdots \oplus L_n$. We call the first chern class of the L_is the i**th chern root**.*

If instead E is a real orientable vector bundle of rank $2n$, then there is a space $\hat{F}(E)$ and a map $\hat{\sigma} : \hat{F}(E) \to E$ such that $\hat{\sigma}^* H_{dR}(E) \to H_{dR}\left(\hat{F}(E)\right)$ is injective and the pullback bundle $\sigma^* E \to \hat{F}(E)$ breaks up into the direct sum of complex line bundles $\sigma^* E = L_1 \oplus \overline{L_1} \oplus \cdots \oplus L_n \oplus \overline{L_n}$.

An idea of the proof is provided in Appendix C. For a more thorough explanation, we refer the reader to [5].

Corollary 8. *Let E be a complex vector bundle of rank n. Then for any $1 \leq k \leq n$, $c_k(E)$ can be written as a polynomial of chern roots.*

This expression of $c(E)$ in terms of the chern roots is incredibly powerful. In fact, it extends beyond just the chern class.

Proposition 22. *If E is a rank n complex vector bundle over M, then*

$$\mathrm{ch}\,(E) = \sum_{i=1}^{n} \exp\left(x_i\,(E)\right) \tag{47}$$

$$\mathrm{Td}\,(E) = \prod_{i=1}^{n} \frac{x_i}{1 - \exp\left(-x_i\right)}\,(E) \tag{48}$$

where $x_i\,(E) = c_1\,(L_i)$ given the decomposition $\sigma^* E = L_1 \oplus \cdots \oplus L_n$.

This is no light fact. In fact, this property is what makes Chern-Weil theory so powerful. We will make use of these formulas in Appendix E.

4.2 Relationship between Chern Class and Euler Class

For this subsection, we assume all vector bundles to be oriented, Euclidean vector bundles.

Proposition 23. *Let ∇^E be a connection on E that is compatible with the Euclidean structure of E. Then $\mathrm{Pf}\left(R^E\right)^2 = \det R^E$.*

Proof. Thinking of R^E as an element of $\mathfrak{so}\,(E)$, by the spectral theorem we can choose a basis $e_1, \ldots, e_n$ of E such that there are real numbers c_j for $1 \leq j \leq n/2$ such that $R^E e_{2j-1} = c_j e_{2j}$ and $R^E e_{2j} = -c_j e_{2j-1}$. As such, we can reduce to the case where E is $\mathbb{R}^2$. In this case, R^E is of the form $\begin{pmatrix} 0 & -c \\ c & 0 \end{pmatrix}$ for some real c, so $R^E e_1 = c e_2$ and $R^E e_2 = -c e_1$. In this case, $\mathrm{Pf}\left(R^E\right) = c$ and $\det R^E = c^2$. $\square$

Corollary 9. *If the rank of E is odd, then $\mathrm{Pf}\left(R^E\right) = 0$.*

Proof. Let A be a skew-symmetric $k \times k$ matrix for k odd. Then,

$$\begin{aligned} \det A &= \det A^\top \\ &= \det\left(-A\right) \\ &= \left(-1\right)^k \det A \end{aligned}$$

$\square$

Theorem 6. *If L is a complex line bundle, then $c_1\,(L) = e\,(L)$.*

Proof. For the chern class we shall consider L to be a complex line bundle and for the Euler class we shall consider L to be a 2-dimensional real bundle. In that case, for the chern class, $R^L = \sqrt{-1}\lambda$ for some number λ, and for the Euler class $R^L = \begin{pmatrix} 0 & -\lambda \\ \lambda & 0 \end{pmatrix}$ for the same number λ. We see that

$$\begin{aligned} c_1\,(L) &= c\,(L) - 1 \\ &= \det\left(1 - \frac{\lambda}{2\pi}\right) - 1 \\ &= -\frac{\lambda}{2\pi} \end{aligned}$$

Similarly,

$$\begin{aligned} e\,(L) &= -\frac{1}{2\pi}\sqrt{\det\begin{pmatrix} 0 & -\lambda \\ \lambda & 0 \end{pmatrix}} \\ &= -\frac{1}{2\pi}\sqrt{\lambda^2} \\ &= -\frac{\lambda}{2\pi} \end{aligned}$$

$\square$

Corollary 10. *For a complex vector bundle E over M, $c_i(E) \in H^{2i}_{dR}(M;\mathbb{Z})$*

Proof. By the Splitting Lemma, it is enough to focus on chern roots. But chern roots are equal to the Euler class of the associated complex line bundle (thought of as an oriented real vector bundle). By the Chern-Gauss-Bonnet theorem, those Euler classes are integers since the Euler characteristic is an integer by definition. Since each c_i is an integral polynomial of the chern roots, they must also be integers. □

Proposition 24. *Let E, F be vector bundles over M. Then $e(E \oplus F) = e(E)\, e(F)$.*

Proof. This is an immediate consequence of Proposition 7. □

Corollary 11. *For a complex vector bundle E of rank n, let $\sigma : F(E) \to E$ be the splitting map. Then the top chern class equals the Euler class.*

Proof.

$$\begin{aligned}
\sigma^* c_n(E) &= c_n(\sigma^* E) \\
&= c_n(L_1 \oplus \cdots \oplus L_n) \\
&= c_1(L_1) \cdots c_1(L_n) \\
&= e(L_1) \cdots e(L_n) \\
&= e(L_1 \oplus \cdots \oplus L_n) \\
&= e(\sigma^* E) \\
&= \sigma^* e(E)
\end{aligned}$$

□

As such, many questions about the Euler class can be answered using polynomials of Chern classes. We conclude this section with a proof of the Gauss-Bonnet Theorem,

Theorem 7 (The Gauss-Bonnet Theorem)**.** *Let M be a 2-dimensional smooth, closed manifold. If g is a metric on M and K its associated Gaussian curvature, then*

$$\int_M K\, dA = 2\pi\chi(M) \tag{49}$$

Proof. By Theorem 3,

$$\begin{aligned}
\chi(M) &= \int_M e(TM) \\
&= \int_M \frac{1}{2\pi} \operatorname{Pf}\left(R^{TM}\right)
\end{aligned}$$

Since TM has a metric g, we know that $R^{TM} = \begin{pmatrix} 0 & -K \\ K & 0 \end{pmatrix} dA$ and that $\operatorname{Pf}\left(R^{TM}\right) = \sqrt{\det R^E}$. As such,

$$\begin{aligned}
2\pi\chi(M) &= \int_M \operatorname{Pf}\left(R^{TM}\right) \\
&= \int_M K\, dA
\end{aligned}$$

□

5 Extensions to Sheaf Cohomology

5.1 Construction of Čech-Cohomology

For the purposes of this paper, we will assume that C is a category and that C is at least a category of abelian groups.

A ***sheaf*** F with values in C on a topological space X assigns to each open set $U \subseteq X$ an element $F(U)$ of the category C such that

1) If $V \subseteq U$ is also an open subset of X, then there is a restriction map $\rho_U^V : F(U) \to F(V)$.

2) If $\alpha, \beta \in F(U)$ are two elements whose restrictions agree on each open set of some cover of U, then $\alpha = \beta$.

3) If an open set U is covered by subsets U_i and the restrictions of elements $\alpha_i \in F(U_i)$ agree on all $U_i \cap U_j$, then there exists an element $\alpha \in F(U)$ that restricts on each U_i to α_i.

What this loose definition is trying to convey is that a sheaf is local data. The more formal definition is given below.

Definition 15. *A **sheaf** on a topological space X modeled on a category C is a contravariant functor $F : \mathsf{Open}(X) \to \mathsf{C}$ such that for every open set $U \subseteq X$ and open cover U_i, for $i \in I$, of U, the sequence*

$$0 \to F(U) \overset{\prod_{i\in I}\rho_U^{U_i}}{\to} \prod_{i\in I} F(U_i) \underset{\prod_{i,j\in I}\rho_{U_j}^{U_i\cap U_j}}{\overset{\prod_{i,j\in I}\rho_{U_i}^{U_i\cap U_j}}{\rightrightarrows}} \prod_{i,j\in I} F(U_i \cap U_j) \tag{50}$$

is exact. What we mean by this is that the map $\prod_{i\in I}\rho_U^{U_i}$ is injective and the images of an element of $\prod_{i\in I} F(U_i)$ coincide in $\prod_{i,j\in I} F(U_i \cap U_j])$ if and only if it is in the image of $F(U)$

There are a few important examples of sheaves.

Example 7. *Given an object A of C, we can give A the discrete topology and consider the sheaf A_X where*

$$A_X(U) := \{\alpha : U \to A : \alpha \text{ continuous}\} \tag{51}$$

*This is called the **locally constant sheaf**. Typically we take A to be a ring or a field.*

Example 8. *Let Y be any topological space. Then we can define the **sheaf of continuous functions***

$$C_X(Y)(U) := \{\alpha : U \to Y : \alpha \text{ continuous}\} \tag{52}$$

Since we do not require the topology on Y to be the discrete topology, we see that, for example, $\mathbb{R}_X$ and $C_X(\mathbb{R})$ are very different sheaves since what qualifies as a continuous function differs.

Example 9. *If X is a complex manifold, we construct the **sheaf of analytic functions***

$$\mathcal{O}_X(U) := \text{the space of holomorphic functions on } U \tag{53}$$

More generally, if (E, X, π) is a holomorphic vector bundle, we define

$$\mathcal{O}_{X,E}(U) := \text{the space of holomorphic sections of } \pi \tag{54}$$

From this definition, we see that $\mathcal{O}_X$ is the space of holomorphic sections of the trivial line bundle. By an abuse of notation, we will often denote $\mathcal{O}_{X,E}$ by $\mathcal{O}_X(E)$.

Definition 16. *Let $\mathcal{U}$ be a well-ordered open cover of X. The* ***space of Čech k-cochains for the open cover*** *$\mathcal{U}$ is*

$$C^k(\mathcal{U},F) := \prod_{(U_0<\cdots<U_k)\in\mathcal{U}^k} F(U_0\cap\cdots\cap U_k) \tag{55}$$

The differential $d^k : C^k(\mathcal{U},F)\to C^{k+1}(\mathcal{U},F)$ is given by

$$d^k(\alpha)(U_0,\ldots,U_{k+1}) := \sum_{i=0}^{k+1}(-1)^i\rho^{U_0\cap\cdots\cap U_{k+1}}_{U_0\cap\cdots\hat{U}_i\cdots\cap U_{k+1}}\alpha\left(U_0\cap\cdots\hat{U}_i\cdots\cap U_{k+1}\right) \tag{56}$$

where the hat over an open subset means that the term is to be omitted. This differential operator defines a ***Čech cochain complex*** *$(C^\bullet(\mathcal{U},F),d^\bullet)$ for the cover $\mathcal{U}$ as*

$$0\to C^0(\mathcal{U},F)\xrightarrow{d^0} C^1(\mathcal{U},F)\to\cdots \tag{57}$$

From this definition, we can construct the ***Čech-cohomology*** of X with respect to our cover $\mathcal{U}$ as

$$\check{H}^k(\mathcal{U},F) := \frac{\ker d^k}{\operatorname{im} d^{k-1}} \tag{58}$$

Unfortunately, this definition is dependent on a choice of cover, which is rather troublesome. We would hope to have a definition of a cohomology which does not depend on a choice of cover. Ideally, we would like to take the injective limit over all covers, partially ordered by refinement. Since a sheaf is local data, we would like to study X at all possible "resolutions".

There is one particular roadblock to taking this injective limit. A priori, we have no obvious map from a Čech cochain with respect to a cover and a Čech cochain with respect to a refinement. Fortunately, this is not really a problem.

Lemma 5. *Let $\tau,\tau' : \mathcal{V}\to\mathcal{U}$ be two refining maps that respect the orders on $\mathcal{U}$ and $\mathcal{V}$. Further assume that $V\subset\tau(V)$ for all $V\in\mathcal{V}$. Then the induced cochain maps τ_*,τ'_* : $C^\bullet(\mathcal{U},F)\to C^\bullet(\mathcal{V},F)$ given by*

$$\tau_*(\alpha)(V_0,\ldots,V_k) = \rho^{V_0\cap\cdots\cap V_k}_{\tau(V_0)\cap\cdots\cap\tau(V_k)}\alpha(\tau(V_0)\cap\cdots\cap\tau(V_k)) \tag{59}$$

(and τ'_* defined similarly) are chain homotopic and induce the same map on cohomology.

Proof. Define $h^k : C^k(\mathcal{U},F)\to C^{k-1}(\mathcal{V},F)$ by

$$h^{k+1}(\alpha)(V_{i_0},\ldots,V_{i_k}) := \sum_{j=0}^{k}(-1)^j\rho\alpha\left(\tau(V_{i_0})\cap\cdots\cap\tau\left(V_{i_j}\right)\cap\tau'\left(V_{i_j}\right)\cap\cdots\cap\tau'(V_{i_k})\right) \tag{60}$$

where ρ stand for the restriction map from $\tau(V_{i_0})\cap\cdots\cap\tau\left(V_{i_j}\right)\cap\tau'\left(V_{i_j}\right)\cap\cdots\cap\tau'(V_{i_k})$ to $V_{i_0}\cap\cdots\cap V_{i_k}$. We will not prove the fact that $\tau_*-\tau'_* = hd+dh$ as there is no insight to be gained by its derivation. □

Definition 17. *If X is a topological space and F a sheaf on X, then the* ***Čech cohomology of X with respect to*** *F is given by*

$$\check{H}^k(X,F) := \lim_{\to}\check{H}^k(\mathcal{U},F) \tag{61}$$

where the injective limit is taken over all ordered open covers, partially ordered by refinement.

The use of injective limits is rather daunting and tedious to work with. Thankfully, there exists a criteria for an open cover to make it "good enough"

Theorem 8 (Leray's cover theorem)**.** *Let $\mathcal{U}$ be a cover of X such that $\check{H}^k(U_0\cap\cdots\cap U_k,F) = 0$ for all $k>0$ and for all finite intersubsections of open sets of the cover. Then the canonical map $\check{H}^\bullet(\mathcal{U},F)\to\check{H}^\bullet(X,F)$ is an isomorphism.*

5.2 Chern Class and Čech Cohomology

Many results from homological algebra, and more specifically cohomology theories, comes from exact sequences. We say that a sequence of sheaves $F \xrightarrow{f} G \xrightarrow{g} H$ is ***exact*** if it is locally exact. This means that for every open subset $U \subseteq X$ and any $x \in U$, for any $\beta \in \ker g$, there exists a neighborhood $V \subseteq U$ of x and $\alpha \in F(V)$ such that $\rho_U^V(\beta) = f(\alpha)$.

Theorem 9 (Leray exact sequence)**.** *Let $0 \to F' \to F \to F'' \to 0$ be an exact sequence. There are then connecting homomorphisms $\delta_k : \check{H}^k(X, F'') \to \check{H}^{k+1}(X, F')$ such that the sequence*

$$\cdots \to \check{H}^{k-1}(X, F'') \to \check{H}^k(X, F') \to \check{H}^k(X, F) \to \check{H}^k(X, F'') \to \check{H}^{k+1}(X, F') \to \cdots \tag{62}$$

is exact.

The Leray exact sequence encodes all the famous long exact sequences for Čech cohomology. For a proof of Theorem 9, see [10].

Let X be a complex manifold and recall that $\mathcal{O}_X$ is the sheaf of analytic functions on X. We denote by $\mathcal{O}_X^*$ the sheaf of non-vanishing analytic functions on X. Let L be an analytic line bundle (a complex line bundle whose transition maps are analytic maps) and $(U_1, \phi_1), (U_2, \phi_2)$ be two intersecting open sets with local trivializations. We know that on each fiber L_x for $x \in U_1 \cap U_2$, if $\phi_2^{-1} \circ \phi_1$ is nonzero, it corresponds to multiplication by some number $M_{21}(x)$. Since we can vary smoothly along fibers, $M_{21}(x)$ is an analytic non-zero function as we vary $x \in U_1 \cap U_2$.

Theorem 10. *For any analytic line bundle (L, π, X) and a local trivialization (U_α, ϕ_α), the functions M_{21} form a 1-cocycle in $C^1(\mathcal{U}, \mathcal{O}_X^*)$. Further, two such cocycles with corresponding analytic line bundles L_1, L_2 determine the same element in $\check{H}^1(X, \mathcal{O}_X^*)$ if and only if L_1 and L_2 are isomoprhic. Finally, if L_1, L_2 are two analytic line bundles corresponding to $\alpha_1, \alpha_2 \in \check{H}^1(X, \mathcal{O}_X^*)$, then $L_1 \otimes L_2$ corresponds to $\alpha_1\alpha_2$.*

With this theorem, we can think of $\check{H}^1(X, \mathcal{O}_X^*)$ as the space of isomorphism classes of analytic line bundles over X. With this in mind, consider the following short exact sequence of sheaves.

$$0 \to \mathbb{Z}_X \to \mathcal{O}_X \overset{f \mapsto e^{2\pi i f}}{\to} \mathcal{O}_X^* \to 0 \tag{63}$$

By Theorem 9, this induces an exact sequence on the coholomogy

$$\begin{aligned} 0 \to \check{H}^0(X, \mathbb{Z}_X) \to \check{H}^0(X, \mathcal{O}_X) \to \check{H}^0(X, \mathcal{O}_X^*) \to \\ \check{H}^1(X, \mathbb{Z}_X) \to \check{H}^1(X, \mathcal{O}_X) \to \check{H}^1(X, \mathcal{O}_X^*) \to \check{H}^2(X, \mathbb{Z}_x) \to \cdots \end{aligned}$$

Proposition 25. *The map $\check{H}^1(X, \mathcal{O}_X^*) \to \check{H}^2(X, \mathbb{Z}_X)$ defined by the sequence above is the chern class.*

For a proof of Proposition 25, see [9] Theorem 4.3.1. The sketch of the proof, which would involve too much algebraic topology to compute here, is that this map behaves exactly like the chern class, and that the chern class is the unique class with those properties.

Definition 18. *A sheaf F on X is **fine** if for every locally finite open cover $\mathcal{U}$ of X, there are homomorphisms $h_U : F(U) \to F(X)$ such that $\sum_{u \in \mathcal{U}} h_U = \mathrm{id}$, and for every $\alpha \in F(U)$, $h_U(\alpha) = 0$ in a neighborhood of $X \setminus U$. These homomorphisms are called **partitions of unity** and are frequently constructed from partitions of unity on X. So, all sheaves for which sections can be multiplied by functions forming a partition of unity are fine (such as sheaves of continuous functions).*

If F is a sheaf on X, a ***fine resolution*** of F is an exact sequence of sheaves

$$0 \to F \to F^0 \xrightarrow{d^0} F^1 \xrightarrow{d^1} F^2 \xrightarrow{d^2} \cdots \tag{64}$$

such that each F^i is fine.

Theorem 11. *If F has a fine resolution F^i and the sequence $F^0 \to F^1 \to \cdots$ is the complex of global sections, then*

$$\check{H}^k(X, F) = \frac{\ker d^k : F^k(X) \to F^{k+1}(X)}{\operatorname{Im} d^{k-1} : F^{k-1}(X) \to F(X)} \tag{65}$$

For a proof of this theorem, see [10] Appendix A7. We will explore two important applications of Theorem 11.

Example 10. *Denote by Ω_X^k the sheaf of k-forms on X. These are all fine sheaves. We then have the fine resolution of $\mathbb{R}_X$:*

$$0 \to \mathbb{R}_X \to \Omega_X^0 \xrightarrow{d} \Omega_X^1 \to \cdots \tag{66}$$

This is called the ***Poincaré resolution****. We see that Theorem 11 then implies that $\check{H}^k(X, \mathbb{R}_X) = H_{dR}^k(X; \mathbb{R})$, which we would expect.*

Example 11. *Let E be an analytic vector bundle over X. Denote by $\Omega_X^{p,q}(E)$ the sheaf of E-valued (p,q)-forms on X. While not obvious, these are all fine sheaves. We then have the fine resolution of $\mathcal{O}_X(E)$:*

$$0 \to \mathcal{O}_X(E) \to \Omega_X^{0,0}(E) \xrightarrow{\overline{\partial}} \Omega_X^{0,1}(E) \to \cdots \to \Omega_X^{0,\dim_{\mathbb{C}} X}(E) \to 0$$

This is called the ***Dolbeault resolution****. We see that Theorem 11 then implies that $\check{H}^q(X, \mathcal{O}_X(E)) = H^{0,q}(X; E)$. More generally, $H^{p,q}(X; E) = \check{H}^q(X, \Omega_X^p(E))$.*

Corollary 12. *If E is an analytic line bundle over X, then $\check{H}^k(X, \mathcal{O}_X(E))$ for all $k > \dim_{\mathbb{C}} X$.*

Much like de Rham cohomology has it's Poincaré duality, Dolbeault cohomology has ***Serre duality***.

Theorem 12. *Let E be an analytic vector bundle over X a compact complex manifold of dimension n. Then $\check{H}^k(X, \mathcal{O}_X(E))$ is dual to $\check{H}^{n-k}\left(X, \mathcal{O}_X(E^*) \otimes \Omega_X^k\right)$*

For a proof of Theorem 12, see [10].

5.3 The Riemann-Roch Theorem

Similar to how we defined the usual Euler characteristic, we define the Euler characteristic of X relative to a vector bundle E as

$$\chi(X, E) = \sum_{i=0}^{\infty} (-1)^i \dim \check{H}^i(X, \mathcal{O}_X(E)) \tag{67}$$

If we take L to be an analytic line bundle over a Riemann surface X, then

$$\chi(X, L) = \dim \check{H}^0(X, \mathcal{O}_X(L)) - \dim \check{H}^1(X, \mathcal{O}_X(L)) + \sum_{i=2}^{\infty} (-1)^i \dim \check{H}^i(X, \mathcal{O}_X(L))$$

By Corollary 12, $\check{H}^i(X, \mathcal{O}_X(L)) = 0$ for $i \geq 2$, so we only need to worry about the first two terms.

Theorem 13 (Riemann-Roch)**.** *Given an analytic line bundle L over a Riemann surface X of genus g,*

$$\chi(X, L) = c_1(L) + 1 - g \tag{68}$$

This is an amazing result. While $\chi(X)$ is purely topological, we have no reason to suspect $\chi(X, L)$ to depend solely on the topology of X. We will spend the remainder of this section proving this theorem.

We will need the following statements. The purpose of these definitions and propositions will all make sense when we see the proof of Theorem 13.

Proposition 26. *Let X be a Riemann surface of genus g. Then* $\dim \check{H}^1(X, \mathcal{O}_X) = g$.

Proof. Consider the short exact sequence of sheaves

$$0 \to \mathbb{C}_X \to \mathcal{O}_X \xrightarrow{d} \Omega_X \to 0$$

This then induces a long exact sequence on cohomology:

$$\begin{aligned} 0 \to \check{H}^0(X, \mathbb{C}_X) \to \check{H}^0(X, \mathcal{O}_X) \to \check{H}^0(X, \Omega_X) \to \\ \check{H}^1(X, \mathbb{C}_X) \to \check{H}^1(X, \mathcal{O}_X) \to \check{H}^1(X, \Omega_X) \to \check{H}^2(X, \mathbb{C}_X) \to \check{H}^2(X, \mathcal{O}_X) \to \cdots \end{aligned}$$

By Corollary 12, $\check{H}^2(X, \mathcal{O}_X) = 0$, so we can end our exact sequence here. It is clear that $\check{H}^0(X, \mathbb{C}_X) = \check{H}^n(X, \mathbb{C}_X) = \mathbb{C}$ by our choice of X. Further, since the only global analytic functions on a compact analytic manifold are constant functions, $\check{H}^0(X, \mathcal{O}_X) = \mathbb{C}$ as well. Serre duality implies that $\check{H}^1(X, \Omega_X)$ is dual to $\check{H}^0(X, \mathcal{O}_X)$, which we have just shown to be $\mathbb{C}$.

The map $\check{H}^0(X, \mathbb{C}_X) \to \check{H}^0(X, \mathcal{O}_X)$ is injective and thus an isomorphism. Similarly, the map $\check{H}^1(X, \Omega_X) \to \check{H}^2(X, \mathbb{C}_X)$ is surjective and thus also an isomorphism. As such, our sequence reduces to

$$0 \to \check{H}^0(X, \Omega_X) \to \check{H}^1(X, \mathbb{C}_X) \to \check{H}^1(X, \mathcal{O}_X) \to 0$$

We then have that $\dim \check{H}^1(X, \mathbb{C}_X) = \dim \check{H}^0(X, \Omega_X) + \dim \check{H}^1(X, \mathcal{O}_X)$. By Serre duality again, we have that

$$\begin{aligned} \dim \check{H}^1(X, \mathbb{C}_X) &= \dim \check{H}^0(X, \Omega_X) + \dim \check{H}^1(X, \mathcal{O}_X) \\ \implies \dim_{\mathbb{R}} \check{H}^1(X, \mathbb{R}_X) &= 2 \dim \check{H}^1(X, \mathcal{O}_X) \\ \implies 2g &= 2 \dim \check{H}^1(X, \mathcal{O}_X) \end{aligned}$$

□

If X is a connected compact Riemann surface, then its complex structure gives an orientation, and thus $\check{H}^2(X, \mathbb{Z}_X) = \mathbb{Z}$. So, $c_1(L)$ for any analytic line bundle L over X is an integer.

Proposition 27. *Let L be an analytic line bundle over a compact Riemann surface X. Let $\sigma \in \Gamma(X, L)$ be a meromorphic section such that σ has a zero or pole on the points $x_1, \ldots, x_k$ with multiplicity $n_1, \ldots, n_k$ (σ has a zero at x_i is $n_i > 0$ and a pole when $n_i < 0$). Then*

$$c_1(L) = \sum_{i=1}^{k} n_i \tag{69}$$

This is a rather involved proof and we direct the eager reader to [10] for a proof.

Proposition 28. *Let L be an analytic line bundle on a compact Riemann surface X. Then L has a meromorphic section, which can be taken to be analytic except at one point.*

Proof. Choose a point $x \in X$ and let $L(nx)$ be the sheaf that assigns to every open subset of X the space of meromorphic sections of L which are analytic except at x, having at most a pole of order n. We have an exact sequence of sheaves

$$0 \to L((n-1)x) \to L(nx) \to \mathbb{C}_X \to 0$$

This sequence induces an exact sequence on cohomology (after some simplifications)

$$0 \to \check{H}^0(X, L((n-1)x)) \to \check{H}^0(X, L(nx)) \to \mathbb{C} \to \check{H}^1(X, L((n-1)x)) \to \check{H}^1(X, L(nx)) \to 0$$

From this, we see that $\dim \check{H}^0(X, L(nx)) = \dim \check{H}^0(X, L((n-1)x)) + 1$ and $\dim \check{H}^1(X, L(nx)) = \dim \check{H}^1(X, L((n-1)x))$. This implies that

$$\begin{aligned}
\chi(X, L(nx)) &= \dim \check{H}^0(X, L(nx)) - \dim \check{H}^1(X, L(nx)) \\
&= \dim \check{H}^0(X, L((n-1)x)) + 1 - \dim \check{H}^1(X, L((n-1)x)) \\
&= \chi(X, L((n-1)x)) + 1
\end{aligned}$$

Now simply choose $n > \chi(X, \mathcal{O}_X(L))$. Then $\chi(X, L(nx)) > 0$, forcing $\dim \check{H}^0(X, L(nx)) > 0$. □

Definition 19. *A **divisor** D on X is a finite sum of points in X with integer weights. The **degree** of a divisor D is the sum of its coefficients.*

*In a similar manner, let f be a meromorphic function on X that is not the zero function. Define $\operatorname{ord}_p f$ to be the order of vanishing of f at p. Then the **divisor** of f is $(f) = \sum_{p \in X} \operatorname{ord}_p f p$.*

If we choose such a divisor $D = \sum n_i x_i$, let $\mathcal{O}_X(D)$ be the sheaf that associates to each open subset $U \subseteq X$ the space of meromorphic functions f on U such that $(f) > -D$.

Corollary 13. *If L is an analytic line bundle over a compact Riemann surface X, if $c_1(L) < 0$, then the only analytic section of L is the zero section.*

Corollary 14. *If L is an analytic line bundle over a compact Riemann surface X, if σ is a section of L that is not the zero section, then $\mathcal{O}_X(L)$ is naturally isomorphic to $\mathcal{O}(-(\sigma))$.*

We are now prepared to prove the Riemann-Roch Theorem. Since $\mathcal{O}_X(L) = \mathcal{O}_X(D)$ for an appropriate choice of divisor D, it is enough to prove the theorem using the sheaf $\mathcal{O}_X(D)$.

Let us first assume that $D = 0$.

$$\begin{aligned}
\chi(X, L) &= \chi(X, \mathcal{O}_X) \\
&= 1 - g \\
&= \deg(0) + 1 - g \\
&= c_1(L) + 1 - g
\end{aligned}$$

Now assume that the theorem is true for some divisor D. We now wish to prove the theorem for $D \pm 1x$. The short exact sequence

$$0 \to \mathcal{O}_X(D) \to \mathcal{O}_X(D + 1x) \to \mathbb{C}_X \to 0$$

induces a long exact sequence on cohomology

$$\begin{gathered}
0 \to H^0(X, \mathcal{O}_X(D)) \to H^0(X, \mathcal{O}_X(D + 1x)) \to \mathbb{C} \to \\
H^1(X, \mathcal{O}_X(D)) \to H^1(X, \mathcal{O}_X(D + 1x)) \to 0
\end{gathered}$$

which gives us that

$$\begin{aligned}\chi(X, D+1x) &= \chi(X, D)+1 \\ &= c_1(\mathcal{O}(D))+2-g \\ &= c_1(\mathcal{O}(D+1x))+1-g\end{aligned}$$

An analogous construction yields $\chi(D-1x) = c_1(\mathcal{O}(D-1x))+1-g$. These statements together give our desired theorem.

In fact, much as how the Chern-Gauss-Bonnet theorem is a stronger version of the Gauss-Bonnet theorem, there is a stronger version of the Riemann-Roch theorem: the Hirzebruch–Riemann–Roch. We offer a proof in Appendix E.

Appendices

A Alternative Approach to Connections and Curvature

Locally, a connection is given by a collection of 1-forms. Let M be n-dimensional manifold and (E, M, π) a k-dimensional real vector bundle over M. Choose a coordinate neighborhood U of M with local coordinates u^i and k smooth sections σ_α of E on U which are linearly independent everywhere. This choice of sections is called a ***local frame field***. Then for every point $p \in U$, $\left\{du^i \otimes \sigma_\alpha : 1 \leq i \leq n, 1 \leq \alpha \leq k\right\}$ forms a basis of $\Lambda^1\left(T_p^* M\right) \otimes E_p$.

If ∇ is a connection on E, then $\nabla \sigma_\alpha \in \Gamma(U, T^*M \otimes E)$. We can then write it as below

$$\nabla \sigma_\alpha = \sum_{1 \leq i \leq n, 1 \leq \beta, k} \Gamma_{\alpha i}^\beta du^i \otimes \sigma_\beta \tag{70}$$

for smooth functions on U $\Gamma_{\alpha i}^\beta$. We then denote by $\omega_\alpha^\beta := \sum_{1 \leq i \leq n} \Gamma_{\alpha i}^\beta du^i$ such that

$$\nabla \sigma_\alpha = \sum_{1 \leq \beta \leq k} \omega_\alpha^\beta \otimes \sigma_\beta \tag{71}$$

Given a local frame field $S = \{\sigma_1, \ldots, \sigma_k\}$, let ω be the matrix whose position in row α and column β is ω_α^β for $1 \leq \alpha, \beta, k$. We call ω the ***connection matrix***. We then can rewrite the Equation 71 as $\nabla S = \omega \otimes S$.

If S' is another local frame field, then we may assume that $S' = AS$ for some matrix $A = \left(a_i^j\right)$ such that $\det A \neq 0$ and the a_i^j are smooth functions on U. If we denote the connection matrix of S' by ω', we see that

$$\begin{aligned}\nabla S' &= dA \otimes S + A \cdot \nabla S \\ &= (dA + A \cdot \omega) \otimes S \\ &= \left(dA \cdot A^{-1} + A \cdot \omega \cdot A^{-1}\right) \otimes S' \\ \implies \omega' &= dA \cdot A^{-1} + A \cdot \omega \cdot A^{-1}\end{aligned}$$

If we had a coordinate covering $\{U, V, W, \ldots\}$, choose a local frame field for each $(S_U, S_V, S_W, \ldots)$ and a connection matrix for each $(\omega_U, \omega_V, \omega_W, \ldots)$. Importantly if, say, U and V intersect, if $S_V = A_{VU} S_U$, for a $k \times k$ matrix of smooth functions on $U \cap V$, then $\omega_V = dA_{VU} \cdot A_{VU}^{-1} + A_{VU} \cdot \omega_U \cdot A_{VU}^{-1}$. Then there is a connection ∇ whose connection matrix on each coordinate covering is the chosen connection matrix. We state without proof that given a connection ∇, and a point $p \in M$, there is a local frame field such that the corresponding ω is zero at p.

If we take an exterior derivative of our transition equation, we see that

$$\begin{aligned} d\left(\omega' \cdot A\right) &= d\left(dA + A \cdot \omega\right) \\ \implies d\omega' \cdot A - \omega' \wedge dA &= dA \wedge \omega + A \cdot d\omega \\ \implies d\omega' \cdot A - \omega' \wedge \left(\omega' \cdot A - A \cdot \omega\right) &= \left(\omega' \cdot A - A \cdot \omega\right) \wedge \omega + A \cdot d\omega \\ \implies \left(d\omega' - \omega' \wedge \omega'\right) \cdot A &= A \cdot \left(d\omega - \omega \wedge \omega\right) \end{aligned}$$

where the wedge products between matrices means that the products of matrix elements are exterior products when the two matrices are multiplied. We define the ***curvature matrix*** $\Omega := \omega - \omega \wedge \omega$. Then our equation above implies that $\Omega' = A \cdot \Omega \cdot A^{-1}$.

If we recall our definition of R^{∇}, we see that

$$\begin{aligned} \nabla\nabla\sigma_\alpha &= \nabla \sum_{1\le\beta\le k} \omega_\alpha^\beta \otimes \sigma_\beta \\ &= \sum_{1\le\beta\le k} \nabla\omega_\alpha^\beta \otimes \sigma_\beta \\ &= \sum_{1\le\beta\le k} d\omega_\alpha^\beta \otimes \sigma_\beta - \omega_\alpha^\beta \nabla\sigma_\beta \\ &= \sum_{1\le\beta\le k} d\omega_\alpha^\beta \otimes \sigma_\beta - \omega_\alpha^\beta \sum_{1\le\gamma\le k} \omega_\beta^\gamma \otimes \sigma_\gamma \\ &= \sum_{1\le\beta\le k} \left(d\omega_\alpha^\beta - \sum_{1\le\gamma\le k} \omega_\alpha^\gamma \omega_\gamma^\beta \right) \otimes \sigma_\beta \\ &= \sum_{1\le\beta\le k} \Omega_\alpha^\beta \sigma_\beta \end{aligned}$$

So the curvature matrix agrees with our definition of R^{∇}.

If $\left(U, u^i\right)$ is a local coordinate system on M, then we construct the Riemannian metric $g := \sum_{1\le i,j\le k} g_{ij} du^i d \otimes du^j$ on M, where g_{ij} is some smooth function on U such that $g_{ij} = g_{ji}$, $\det g_{ij}\left(p\right) \neq 0$ for all $p \in M$, and $g\left(X, X\right) \geq 0$ for all $X \in T_pM$ and $p \in U$. If we examine our definition of a metric-compatible connection, we see that $\nabla g = 0$ is equivalent to

$$\begin{aligned} 0 &= \sum_{1\le i,j,l\le k} \left(dg_{ij} - \omega_i^l g_{lj} - \omega_j^l g_{il}\right) \otimes du^i \otimes du^j \\ \implies dg_{ij} &= \sum_{1\le l\le k} \omega_i^l g_{lj} + \omega_j^l g_{il} \\ \implies dg &= \omega \cdot g + g \cdot \omega^\top \end{aligned}$$

If we take another exterior derivative, we find that

$$\begin{aligned} 0 &= d\omega \cdot g - \omega \wedge dg + dg \wedge \omega^\top + g \wedge \left(d\omega\right)^\top \\ &= \left(d\omega - \omega \wedge \omega\right) \cdot g + g \cdot \left(d\omega - \omega \wedge \omega\right)^\top \\ &= \Omega \cdot g + \left(\Omega \cdot g\right)^\top \end{aligned}$$

If we denote by $\Omega_{ij} = \Omega_i^k g_{jk}$, then the above equation becomes $\Omega_{ij} + \Omega_{ji} = 0$, so Ω is skew-adjoint[9].

If E chosen to be the tangent bundle, and ∇ the Levi-Civita connection, then it can be shown that $\Omega_{ij} = \sum_{1\le k,l\le \dim M} \frac{1}{2} R_{ijkl} du^k \wedge du^l$, where R_{ijkl} are the components of the

[9]This technique works for general bundle metrics, but it is easiest to see what is happening in the Riemannian metric case.

Riemannian curvature 4-tensor. If M is 2-dimensional, then we see that by the symmetries of the Riemannian curvature 4-tensor

$$\begin{aligned}\Omega_{11} &= 0 \\ \Omega_{12} &= R_{1212} du^1 \wedge du^2 \\ \Omega_{21} &= -R_{1212} du^1 \wedge du^2 \\ \Omega_{22} &= 0\end{aligned}$$

Up to sign, $R_{1212} = K \det g$, where K is the Gaussian curvature of M. Since $\det g du^1 \wedge du^2 = dA$, the area differential, we see that, $\Omega = \begin{pmatrix} 0 & -K \\ K & 0 \end{pmatrix} dA$.

B A Brief Venture into K-Theory

Let M be an abelian monoid. A ***group completion*** of M is an abelian group $M^{-1}M$ together with a monoid map $[\,] : M \to M^{-1}M$ such that for every abelian group A, and a monoid map $\phi : M \to A$, there is a unique abelian group homomorphism $\tilde{\phi} : M^{-1}M \to A$ such that $\tilde{\phi}[m] = \phi(m)$ for all $m \in M$. In a similar manner, let R be a semi-ring. Then we can apply group completion on the additive structure of R to construct a ring $R^{-1}R$.

Semi-rings that are very important to us are $\mathbf{VB}_{\mathbb{R}}(X)$ and $\mathbf{VB}_{\mathbb{C}}(X)$, the set of real and complex vector bundles over a space X, respectively. We see that this is a semi-ring since the Whitney sum of vector bundles over X is still a vector bundle over X of the same type, as is the tensor product of vector bundles. We denote the group completions of $\mathbf{VB}_{\mathbb{R}}(X)$ and $\mathbf{VB}_{\mathbb{C}}(X)$ by $KO(X)$ and $KU(X)$. When it is understood that we will be working with just real vector bundles or just complex vector bundles, we will simply right $K(X)$.

Let $f : X \to Y$ be a continuous map between spaces. Then for every vector bundle E over Y, we have a vector bundle f^*E over X. We can then think of f^* as a map from $\mathbf{VB}(Y) \to \mathbf{VB}(X)$. Importantly, this map is a morphism of moniods and semi-rings, so it induces a map $f^* : K(Y) \to K(X)$. Importantly, this new map f^* depends only on the homotopy class of f in $C^0(X, Y)$.

Consider the chern character class as a map $\mathrm{ch} : \mathbf{VB}_{\mathbb{C}}(X) \to \check{H}^*(X, \mathbb{Q}_X)$. This is a monoid map, so there is an induced abelian group homomorphism $\tilde{ch} : KU(X) \to \check{H}^*(X, \mathbb{Q}_X)$. A quick computation shows that $\tilde{ch}$ is actually a ring homomorphism.

Theorem 14. *If X is a compact manifold, then*

$$\tilde{\mathrm{ch}} : KU(X) \otimes \mathbb{Q} \cong \check{H}(X, \mathbb{Q}_X) \tag{72}$$

Let X be compact space and Y a closed subspace. We would like to construct $KU(X, Y)$, the group completion of $\mathbf{VB}_{\mathbb{C}}(X)$ with Y identified. If Y is the empty set, we will define $KU(X, Y) := KU(X)$. If Y is a single point, then let $\iota : Y \to X$ be the inclusion map and define $K(X, Y) := \ker \iota^* : K(X) \to K(Y)$. Otherwise, if CY denotes the cone of Y with vertex v, define $K(X, Y) := K(X \cup CY, v)$. As before, if $f : (X, Y) \to (X', Y')$, i.e. $f(X) \subseteq X'$ and $f(Y) \subseteq Y'$, then there is an induced map $f^*K(X', Y') \to K(X, Y)$ again depending only on the homotopy class of f.

If $i : (Y, \varnothing) \to (X, \varnothing)$ and $j : (X, \varnothing) \to (X, Y)$, then we have a short exact sequence of $K(X)$-modules

$$0 \to K(X, Y) \xrightarrow{j^*} X(X) \xrightarrow{i^*} K(Y) \to 0 \tag{73}$$

Further, if $f : X \to Y$ is a map such that $f \circ i$ is the identity, then we have the split exact sequence

$$0 \to K(X, Y) \xrightarrow{j^*} X(X) \underset{f^*}{\overset{i^*}{\rightleftarrows}} K(Y) \to 0 \tag{74}$$

Now assume that E, F are complex line bundles over X and that $\alpha : E\big|_Y \to F\big|_Y$ is an isomorphism. Define the following

$$
\begin{aligned}
I_0 &:= (0,1] & I_1 &:= [0,1) \\
Z_0 &:= X \times \{0\} \cup Y \times I_1 & Z_1 &:= X \times \{1\} \cup Y \times I_0 \\
E_0 &:= F & E_1 &:= E
\end{aligned}
$$

Further, let $f_0 : Z_0 \to X$, $f_1 : Z_1 \to X$, and $f : Z := X \times \{0\} \cup X \times \{1\} \cup Y \times \{1\} \to X$ be induced by the projection map $X \times I \to X$. Then for $i = 0, 1$, $f_i^* (E_i)$ is a bundle over Z_i and α induces an isomorphism $f_1^* (E_1) \to f_0^* (E_0)$ on $Z_0 \cap Z_1$. This defines a complex vector bundle L over Z.

The element $L - f^* F$ in $K(Z)$ is trivial when restricted to $X \times \{0\}$. Thinking of f as a map from Z to $X \times \{0\}$, we have the split exact sequence

$$0 \to K(Z, X \times \{0\}) \to K(Z) \underset{f^*}{\rightleftarrows} K(X \times 0) \to 0 \tag{75}$$

So, $L - f^* F$ defines an element of $K(Z, X \times \{0\}) = K(X, Y)$. We denote this element by $d(E, F, \alpha)$ and call it the ***difference bundle*** in $K(X, Y)$.

Theorem 15. *The difference bundle has the following properties*

1) *If $f : (X, Y) \to (X', Y')$ is a map, then $d(f^* E, f^* F, f^* \alpha) = f^* d(E, F, \alpha)$*
2) *$d(E, F, \alpha)$ depends only on the homotopy class of α*
3) *If $Y = \varnothing$, then $d(E, F, \alpha) = E - F$*
4) *If $j^* : K(X, Y) \to K(X)$, then $j^* d(E, F, \alpha) = E - F$*
5) *$d(E, F, \alpha) = 0$ if and only if there is a vector bundle G over X such that $\alpha \oplus 1$ extends to an isomorphism $E \oplus G \to F \oplus G$ over all of X*
6) *$d(E_1 \oplus E_2, F_1 \oplus F_2, \alpha_1 \oplus F_2) = d(E_1, F_1, \alpha_1) + d(E_2, F_2, F_2)$*
7) *$d(E, F, \alpha) + d\left(E, F, \alpha^{-1}\right) = 0$*
8) *If $\beta : F \to G$ is an isomorphism on $F\big|_Y \to G\big|_Y$, then $d(E, G, \beta\alpha) = d(E, F, \alpha) + d(F, G, \beta)$*

Let W be a real vector bundle or rank $2 \dim M$ for compact M with a bundle metric. Let $B(W)$ and $S(W)$ be the disc and sphere bundles of W and $\pi : B(W) \to M$ the projection map. Consider the difference bundle $d(\pi^* E, \pi^* F, \alpha)$ for some isomorphism $\alpha : \pi^* E\big|_{S(W)} \to \pi^* F\big|_{S(W)}$. The chern characteristic class then defines an element

$$\mathrm{ch}(d(\pi^* E, \pi^* F, \alpha)) \in H\left(B(W), S(W), \mathbb{Q}_{B(W)}\right) \tag{76}$$

Recall the Thom isomorphism $T : H^\bullet(M, \mathbb{Q}) \to H^{\bullet + 2\dim M}$ is given by $T(\alpha) = \pi^* \alpha \wedge \Phi(B(W))$. Let $j : (B(W), \varnothing) \to (B(W), S(W))$. We then notice that $j^* \Phi(W) = \pi^* e(W)$. It then follows that $j^* T(\alpha) = \pi^* (\alpha \wedge e(W))$.

Theorem 16. *Let E, F be complex vector bundles over X compact and let W be a real oriented vector bundle over X. Let $B(W)$ and $S(W)$ be the unit disc and sphere bundles of W, $\pi : B(W) \to X$ the projection map, and $\alpha : \pi^* E\big|_{S(W)} \to \pi^* F\big|_{S(W)}$ an isomorphism. Then*

$$e(W) \wedge T^{-1} \mathrm{ch}(d(\pi^* E, \pi^* F, \alpha)) = \mathrm{ch}(E) - \mathrm{ch}(F) \tag{77}$$

Proof. We see that

$$\begin{aligned} j^* \operatorname{ch}(d(\pi^*E, \pi^*F, \alpha)) &= \operatorname{ch}(j^* d(\pi^*E, \pi^*F, \alpha)) \\ &= \operatorname{ch}(\pi^*E - \pi^*F) \\ &= \pi^* \operatorname{ch}(E - F) \end{aligned}$$

As such,

$$\begin{aligned} \pi^* \operatorname{ch}(E - F) &= j^* \operatorname{ch}(d(\pi^*E, \pi^*F, \alpha)) \\ &= j^* T T^{-1} \operatorname{ch}(d(\pi^*E, \pi^*F, \alpha)) \\ &= \pi^* \left(e(W) \wedge T^{-1} \operatorname{ch}(d(\pi^*E, \pi^*F, \alpha))\right) \end{aligned}$$

Since π^* is an isomorphism, this implies $e(W) \wedge T^{-1} \operatorname{ch}(d(\pi^*E, \pi^*F, \alpha)) = \operatorname{ch}(E - F)$. □

C Projective Vector Bundles

Let V be a complex vector space and $P(V) := \{1 - \text{dimensional subspaces of } V\}$. On $P(V)$, we will consider three vector bundles: the ***product bundle*** $\hat{V} := P(V) \times V$, the ***universal subbundle*** $S := \left\{(\ell, v) \in \hat{V} : v \in \ell\right\}$, and the ***universal quotient subbundle*** Q, which is defined by the exact sequence $0 \to S \to \hat{V} \to Q \to 0$. We call this sequence the ***tautological exact sequence over*** $P(V)$. We call S^*, the dual of S, the ***hyperplane bundle***.

Now consider a complex vector bundle (E, π, M) of rank k. The projectivization of E, denoted $(P(E), \varpi, M)$, is defined such that $P(E)_x = P(E_x)$. So, a point in $P(E)$ is a line in E_p. Just as before, there are many natural vector bundles over $P(E)$ we can construct. The main bundles that we will consider are ϖ^*E (the pullback bundle of E over $P(E)$), the universal subbundle $S := \{(\ell, v) \in \varpi^*E : v \in \ell\}$, and the universal quotient bundle Q determined by the exact sequence $0 \to S \to \varpi^*E \to Q \to 0$.

Since S is a line bundle over $P(E)$, so too is S^*. Let $x = c_1(S^*)$. Then x defines a cohomology class in $H^2_{dR}(P(E))$. By the naturality property of c_1, the restriction of S to a fiber $P(E_p)$ is the universal subbundle $\tilde{S}$ of $P(E_p)$ and $c_1\left(\tilde{S}\right)$ is the restriction of x to $P(E_p)$. Thus, the cohomology classes $1, x, \ldots, x^{k-1}$ are global classes on $P(E)$ that when restricted to the fibers generate the cohomology of the fiber. By Theorem 5, $H^*(P(E))$ is a free module over $H^*(M)$ with basis $\left\{1, x, \ldots, x^{k-1}\right\}$. So, x^k can be written uniquely as a linear combination of the basis elements with coefficients in $H^*(M)$. These coefficients are the i-th chern classes that we have already constructed.

$$0 = x^k + \varpi^* c_1(E)\, x^{k-1} + \cdots + \varpi^* c_k(E) \tag{78}$$

In other words

$$H^*(P(E)) = H^*(M)[x] / \langle x^k + \varpi^* c_1(E)\, x^{k-1} + \cdots + \varpi^* c_k(E) \rangle \tag{79}$$

Theorem 17. *Let (E, M, π) be a complex line bundle of rank k. There is a space $F(E)$ and a map $\sigma : F(E) \to M$ such that $\sigma^*E = L_1 \oplus \cdots \oplus L_k$ where L_i are each line bundles, and σ^* embeds $H^*(M)$ into $H^*(F(E))$.*

Proof. First note that if $0 \to A \to B \to C \to 0$ is an exact sequence of smooth complex line bundles, then $B \cong A \oplus C$ as a smooth bundle. We will prove the theorem by induction.

If $k = 1$, then we are done. If $k = 2$, let $F(E) = P(E)$. Then $0 \to S_E \to \sigma^*E \to Q_E \to 0$ is an exact sequence and $\sigma^*E = S_E \oplus Q_E$. The second half is a consequence of our previous work.

If $k = 3$, then we still have the short exact sequence $0 \to S_E \to \varpi^*E \to Q_E \to 0$ (where $\varpi : P(E) \to M$ is the bundle map of $P(E)$). The quotient bundle Q_E is now a rank 2 vector bundle over $P(E)$ with bundle map ρ. But we know how to split a rank-2 bundle into a direct sum of line bundles. Consider the diagram below

$$\begin{tikzcd}
& & \rho^* S_E \oplus S_{Q_E} \oplus Q_{Q_E} \arrow[d] \\
& S_E \oplus Q_E \arrow[d] & P\left(Q_E\right) \arrow[dl, "\varrho"] \\
E \arrow[d] & P\left(E\right) \arrow[dl, "\varpi"] & \\
M & &
\end{tikzcd}$$

Then, taking $P\left(Q_E\right)$ to be $F\left(E\right)$ solves the first half of the theorem. If we let $x_1 = \varrho^* c_1\left(S_E^*\right)$ and $x_2 = c_1\left(S_{Q_E}^*\right)$, then with some abuse of notation

$$H^*\left(F\left(E\right)\right) = H^*\left(M\right)\left[x_1, x_2\right] / \langle x_1^3 + c_1\left(E\right) x^2 + c_2\left(E\right) x + c_3\left(E\right), x_2^2 + c_1\left(Q_E\right) x_2 + c_2\left(Q_E\right)\rangle$$

From this process it is now clear that for finite k, one can construct a diagram similar to that below

$$\begin{tikzcd}
& & & S_1 \oplus S_2 \oplus \cdots S_{k-1} \oplus Q_{k-1} \arrow[d] \\
& & S_1 \oplus S_2 \oplus Q_2 \arrow[d] & P\left(Q_{k-2}\right) =: F\left(E\right) \arrow[dl, dashed] \\
& S_1 \oplus Q_1 & P\left(Q_1\right) \arrow[dl] & \\
E \arrow[d] & P\left(E\right) \arrow[dl] & & \\
M & & &
\end{tikzcd}$$

where the dashed line is to show an arrow extending along multiple implied components of the diagram. We then also have that $H^*\left(E\left(E\right)\right)$ is a free $H^*\left(E\right)$ module having a baiss of all monomials of the form

$$x_1^{a_1} x_2^{a_2} \cdots x_{k-1}^{a_{k-1}} \tag{80}$$

for $0 \leq a_i \leq k - i$ and $x_i = c_i\left(S_i^*\right)$. $\square$

Corollary 15 (The Splitting Principle)**.** *To prove a polynomial identity in the Chern classes of complex vector bundles, it suffices to prove it under the assumption that the vector bundles are direct sums of line bundles.*

D A Brief Survey of Complex Manifolds

D.1 Construction of Complex Manifolds

Definition 20. *A* ***complex manifold of dimension*** *n is a topological space M with a cover $\mathcal{U} = \{U_\alpha\}$ and charts $\phi_\alpha : U_\alpha \to \mathbb{C}^n$ such that $U_\alpha, U_\beta \in \mathcal{U}$ then $\phi_{\beta\alpha} := \phi_\beta \circ \phi_\alpha^{-1}$ is holomorphic. We call the collection (U_α, ϕ_α) a* ***holomorphic structure****.*

Definition 21. *Let E be a real vector space of dimension $2n$. The following are equivalent definitions of a* ***complex structure****:*

1) an $\mathbb{R}$-linear map $J : E \to E$ such that $J^2 = -\mathrm{Id}$

2) a complete vector space F and an $\mathbb{R}$-isomorphism $f : E \to F$

3) a complete subspace $K \subset \mathbb{C} \otimes_{\mathbb{R}} E$ of complex dimension n such that $K \cap \overline{K} = \{0\}$

If M is a smooth manifold of dimension $2n$, we call a collection of complex structures on each tangent space an ***almost complex structure***. We call the collection (M, J) an ***almost complex manifold***. If M is a complex manifold, then there is an obvious almost complex structure on M since the tangent spaces of M are complex vector spaces. We will call such an almost complex structure a ***natural complex structure***.

If $K \subset \mathbb{C} \otimes_{\mathbb{R}} E$ defines a complex structure on E, then $\mathbb{C} \otimes_{\mathbb{R}} E = K \oplus \overline{K}$. There are natural isomorphisms $p_K : E \to K$ and $p_{\overline{K}} : E \to \overline{K}$ such that $1 \otimes x = p_K(x) + p_{\overline{K}}(x)$.

In a similar vein, an almost-complex structure induces a grading on the tangent bundle. If (M, J) is an almost complex manifold, let $T^{1,0}M$ be the eigenbundle of $\mathbb{C} \otimes TM := TM^{\mathbb{C}}$ corresponding to the eigenvalue i of J. Similarly, let $T^{0,1}M$ be the eigenbundle of $\mathbb{C} \otimes TM := TM^{\mathbb{C}}$ corresponding to the eigenvalue $-i$ of J.

Proposition 29. *If (M, J) is an almost complex manifold,*

$$T^{1,0}M = \{X - iJX : X \in TM\} \tag{81}$$
$$T^{0,1}M = \{X + iJX : X \in TM\} \tag{82}$$
$$TM^{\mathbb{C}} = T^{1,0}M \oplus T^{0,1}M \tag{83}$$

Proof. If $X \in TM^{\mathbb{C}}$, let $X_1 = \frac{1}{2}(X - i \cdot JX)$ and $X_2 = \frac{1}{2}(X + i \cdot JX)$. We immediately have that $X = X_1 + X_2$. Further,

$$\begin{aligned} JX_1 &= J\frac{1}{2}(X - i \cdot JX) \\ &= \frac{1}{2}(JX + iX) \\ &= \frac{i}{2}(-i \cdot JX + X) \\ &= iX_1 \end{aligned}$$

A similar calculation shows that X_2 has eigenvalue $-i$. So, $X_1 \in T^{1,0}M$ and $X_2 \in T^{0,1}M$.

Alternatively, every element of $TM^{\mathbb{C}}$ can be written as $Z = X + iY$ for $X, Y \in TM$. If Z is actually in $T^{1,0}M$, multiplication by J yields that $JX = -Y$ and $JY = X$, so $Z = X - iJX$. $\square$

Now consider $\Lambda^*_{\mathbb{C}}(T^*M) := \mathbb{C} \otimes \Lambda^*(T^*M)$ for an almost complex manifold (M, J). We can think of elements of $\Lambda^*_{\mathbb{C}}(T^*M)$ as either complex valued forms or the composition $\alpha + i\beta$ for α, β real forms. $\Lambda^1_{\mathbb{C}}(T^*M)$ has a natural grading as

$$\Lambda^{1,0}_{\mathbb{C}}(T^*M) := \left\{\alpha \in \Lambda^1_{\mathbb{C}}(T^*M) : \alpha(Z) = 0, \forall Z \in T^{0,1}M\right\} \tag{84}$$
$$\Lambda^{0,1}_{\mathbb{C}}(T^*M) := \left\{\alpha \in \Lambda^1_{\mathbb{C}}(T^*M) : \alpha(Z) = 0, \forall Z \in T^{1,0}M\right\} \tag{85}$$

Proposition 30. *If (M, J) is an almost complex manifold,*

$$\Lambda^{1,0}M = \left\{\alpha - i\alpha \circ J : \alpha \in \Lambda^1(T^*M)\right\} \tag{86}$$
$$\Lambda^{0,1}M = \left\{\alpha + i\alpha \circ J : \alpha \in \Lambda^1(T^*M)\right\} \tag{87}$$
$$\Lambda^1_{\mathbb{C}}(T^*M) = \Lambda^{1,0}(T^*M) \oplus \Lambda^{0,1}(T^*M) \tag{88}$$

We can define $\Lambda^{p,0}(T^*M)$ and $\Lambda^{0,q}(T^*M)$ in similar manners. We denote by $\Lambda^{p,q}(T^*M) := \Lambda^{p,0}(T^*M) \otimes \Lambda^{0,q}(T^*M)$. Then we see that $\Lambda^k_{\mathbb{C}}(T^*M) = \bigoplus_{p+q=k} \Lambda^{p,q}(T^*M)$. We call elements of $\Lambda^k_{\mathbb{C}}(T^*M)$ (p,q)-forms.

If J is a natural complex structure, then the spaces $\Omega^{p,q}(M) := \Gamma(M, \Lambda^{p,q}(T^*M))$ are well defined and that $\Omega^k(M) = \bigoplus_{p+q=k} \Omega^{p,q}(M)$. We can also extend the exterior derivative to a $\mathbb{C}$-linear map, which then decomposes into $d = \partial + \overline{\partial}$ where $\partial : \Lambda^{p,q}(T^*M) \to \Lambda^{p+1,q}(T^*M)$ and $\overline{\partial} : \Lambda^{p,q}(T^*M) \to \Lambda^{p,q+1}(T^*M)$. If we examine the property $d^2 = 0$, we see that

$$\begin{aligned} 0 &= d \circ d \\ &= \left(\partial + \overline{\partial}\right) \circ \left(\partial + \overline{\partial}\right) \\ &= \partial^2 + \overline{\partial}\partial + \partial\overline{\partial} + \overline{\partial}^2 \end{aligned}$$

Since $\partial^2, \overline{\partial}\partial + \partial\overline{\partial}, \overline{\partial}^2$ are all maps to different spaces, we see that they all must be zero maps. We say that a $(p,0)$-form α is ***holomorphic*** if $\overline{\partial}\alpha = 0$. We say that a map $f : M \to N$ for two complex manifolds M, N is ***holomorphic*** if $f_* T_p M^{1,0} \subseteq T_{f(p)} M^{1,0}$. Since the decomposition of $TM^{\mathbb{C}}$ is preserved under holomorphic maps, so too is the decomposition of $\Omega^k(M)$. As such, if $f : M \to N$ is holomorphic, then $f^*\Omega^{p,q}(N) \subseteq \Omega^{p,q}(M)$ and $\overline{\partial} \circ f^* = f^* \circ \overline{\partial}$.

These properties let us define the ***Delbeault cohomology groups*** of a complex manifold M as

$$H^{p,q}(M) := \frac{\overline{\partial}\text{-closed } (p,q)\text{-forms}}{\overline{\partial}\text{-exact } (p,q)\text{-forms}} \tag{89}$$

If $f : M \to N$ is holomorphic, then we see that $f^* H^{p,q}(N) \to H^{p,q}(M)$ is a homomorphism.

Proposition 31 ($\overline{\partial}$-Poincaré Lemma)**.** *If $\alpha \in \Omega^{1,0}(M)$ is $\overline{\partial}$-closed, then it is locally $\overline{\partial}$-exact.*

Proposition 32. *Let $\alpha \in \Omega^{1,1}(M) \cap \Omega^2(M)$ be a real 2-form on M. Then α is d-closed if and only if for ever point $x \in M$, there is a neighborhood U and a function $u : U \to \mathbb{R}$ such that $\alpha\big|_U = i\partial\overline{\partial}u$.*

D.2 Holomorphic Vector Bundles

Definition 22. *Let M be a complex manifold and E a complex vector bundle over M. We say that E is a **holomorphic vector bundle** if the transition maps of the local trivialization can be taken to be holomorphic maps.*

Important examples of holomorphic vector bundles are TM and $\Lambda^{p,0}(T^*M)$ for a complex manifold M.

If E is a holomorphic vector bundle of rank k, we define $\Omega^{p,q}(M,E) := \Gamma(M, \Lambda^{p,q}(T^*M) \otimes E)$. There is then a natural map $\overline{\partial} : \Omega^{p,q}(M,E) \to \Omega^{p,q+1}(M,E)$ constructed as follows. Let $\alpha \in \Omega^{p,q}(M,E)$ be given by $(\alpha_1, \ldots, \alpha_k)$ in some local trivialization (a k-vector of complex valued (p,q)-forms). We then define $\overline{\partial}\alpha := \left(\overline{\partial}\alpha_1, \ldots, \overline{\partial}\alpha_k\right)$. If α is given by $(\beta_1, \ldots, \beta_k)$ in some other local trivialization, then we see that $\beta_i = \sum_{j=1}^k g_{ij}\alpha_j$ for some collection of holomorphic functions $g_{ij} : M \to \mathbb{C}$. As such, $\overline{\partial}\beta_i = \sum_{j=1}^k g_{ij}\overline{\partial}\alpha_j$, so $\overline{\partial}\alpha$ does not depend on a choice of local trivialization. We see that this map satisfies $\overline{\partial}^2 = 0$ and the Leibniz rule.

Definition 23. *If $\overline{\partial} : \Omega^{p,q}(M,E) \to \Omega^{p,q+1}(M,E)$, for some complex vector bundle E, satisfies the Leibniz rule, we call $\overline{\partial}$ a **pseudo-holomorphic structure**. If it also satisfies $\overline{\partial}^2 = 0$, then we call it a **holomorphic structure**. A form $\alpha \in \Omega^{p,q}(M,E)$ of a pseudo-holomorphic vector bundle $\left(E, \overline{\partial}\right)$ is called **holomorphic** if $\overline{\partial}\sigma = 0$.*

Theorem 18. *A complex vector bundle E is holomorphic if and only if it has a holomorphic structure $\overline{\partial}$.*

Now assume that E is a complex vector bundle over a complex manifold M. Since ∇^E is a choice of extension of d to bundle-valued forms, we should expect there to also be a grading of our connections. If $\pi^{1,0} : \Lambda^1_{\mathbb{C}}(T^*M) \to \Lambda^{1,0}(T^*M)$ and $\pi^{0,1} : \Lambda^1_{\mathbb{C}}(T^*M) \to \Lambda^{0,1}(T^*M)$ are projections, then we define $\nabla^{1,0} := \pi^{1,0} \circ \nabla^E$ and $\nabla^{0,1} := \pi^{0,1} \circ \nabla^E$[10]. We can extend these to maps $\nabla^{1,0} : \Omega^{p,q}(M,E) \to \Omega^{p+1,q}(M,E)$ and $\nabla^{0,1} : \Omega^{p,q}(M,E) \to \Omega^{p,q+1}(M,E)$ satisfying

$$\nabla^{1,0}(\alpha \otimes \sigma) = \partial\alpha \otimes \sigma + (-1)^{\deg\alpha}\,\alpha \wedge \nabla^{1,0}\sigma \tag{90}$$

$$\nabla^{0,1}(\alpha \otimes \sigma) = \overline{\partial}\alpha \otimes \sigma + (-1)^{\deg\alpha}\,\alpha \wedge \nabla^{0,1}\sigma \tag{91}$$

for $\alpha \in \Omega^{p,q}(M)$ and $\sigma \in \Gamma(M,E)$. We see that $\nabla^{0,1}$ is a pseudo-holomorphic structure on E.

Since every connection ∇ has a grading, so too does every curvature R^E. We see that

$$\begin{aligned}
\left(R^E\right) &= \left(\nabla^E\right)^2 \\
&= \left(\nabla^{1,0} + \nabla^{0,1}\right)^2 \\
&= \left(\nabla^{1,0}\right)^2 + \left(\nabla^{1,0}\nabla^{0,1} + \nabla^{0,1}\nabla^{1,0}\right) + \left(\nabla^{0,1}\right)^2 \\
&:= R^{2,0} + R^{1,1} + R^{0,2}
\end{aligned}$$

As such, we see that if $R^{0,2} = 0$, then $\nabla^{0,1}$ is a holomorphic structure on E, so E is a holomorphic vector bundle.

Definition 24. *A* ***Hermitian structure*** *H on a complex vector bundle E is a collection of Hermitian products that varies smoothly along fibers. In other words, for every $x \in M$, the restriction $H : E_x \to E_x$ satisfies*

1) *$H(u,v)$ is $\mathbb{C}$-linear in U*
2) *$H(u,v) = \overline{v,u}$*
3) *$H(u,u) \leq 0$ for all $u \neq 0$*
4) *$H(u,v)$ is smooth for every section $u,v \in \Gamma(M,E)$.*

It is clear that every complex vector bundle admits a Hermitian structure. We can split this object into real and imaginary parts $H(u,v) = F(u,v) + iG(u,v)$. Property 2) says that $F(u,v) = F(v,u)$ and that $G(u,v) = -G(v,u)$. Since F, G are bilinear forms, their symmetry conditions imply that F behaves like a real metric and G like a 2-form. If each E_x has a complex structure, then we see that $H(Ju,v) = iH(u,v)$. This implies that $F(Ju,JV) = F(u,v)$ and $G(Ju,JV) = G(u,v)$.

Given a connection ∇^E for a Hermitian vector bundle (E,H), we say that ∇^E is an H-connection if ∇^E is compatible with H.

Theorem 19. *For every Hermitian structure H on $\left(E,\overline{\partial}\right)$ a holomorphic vector bundle, there exists a unique H-connection ∇ such that $\nabla^{1,0} = \overline{\partial}$. We call this connection the* ***Chern connection****.*

D.3 Hermitian and Kähler Manifolds

Definition 25. *Let (M,J) be an almost complex manifold. A* ***Hermitian metric*** *on M is a Riemannian metric h such that $h(X,Y) = h(JX,JY)$ for all $X,Y \in TM$. The fundamental form of h is defined as $\omega(X,Y) := h(JX,Y)$. We will also denote the extension of h to $TM^{\mathbb{C}}$ using $\mathbb{C}$-linearity by h.*

If M is a complex manifold, TM is a complex vector bundle. If h is a Hermitian metric on M, then $H(X,Y) := h(X,Y) - ih(JX,Y)$ is a Hermitian structure on (TM,J). Similarly, if H is a Hermitian structure on TM, then $\mathrm{Re}(H)$ is a Hermitian metric on M.

[10]For space and ease of notation, we drop the superscript E on the graded connection since the choice of connection is understood

Remark 3. *Every almost complex manifold admits a Hermitian metric. If g is a Riemannian metric on M, then $h(X,Y) := g(X,Y) + g(JX,JY)$*

Definition 26. *Given an almost complex manifold (M,J) with Hermitian metric h, we say that h is* ***Kähler*** *if J is a natural complex structure and $d\omega = 0$. Since ω is a real form of type $(1,1)$, it can be written locally as $i\partial\overline{\partial}u$ for some real function u. We call u the* ***local Kähler potential*** *of h.*

Theorem 20. *A Hermitian metric h on an almost complex manifold (M,J) is Kähler if and only if about each point in M, there are holomorphic coordinates such that up to second order, h looks like the standard Hermitian product.*

Proposition 33. *If ∇ is the Levi-Civita connection on an almost complex manifold (M,J) with Hermitian metric h, the Hermitian structure on TM is equal to $\overline{\partial}$, the anti-holomorphic component of the exterior derivative.*

Theorem 21. *A Hermitian metric h on an almost complex manifold (M,J) is Kähler if and only if the Levi-Civita connection on M is also the Chern connection.*

We call the collection (M,J,h) of an almost complex manifold with Hermitian metric H an ***almost Hermitian manifold***.

Recall Section 1.3 the Hodge-star operator $\star$, $\langle\cdot,\cdot\rangle_\Omega$, and the formal adjoint of the exterior derivative d^* with respect to a Riemannian metric.

Just as d had a decomposition into $\partial+\overline{\partial}$, $d^* = \partial^* + \overline{\partial}^*$, where $\partial^* = -\star\overline{\partial}\star : \Omega^{p,q}(M) \to \Omega^{p-1,q}(M)$ and $\overline{\partial}^* = -\star\partial\star : \Omega^{p,q}(M) \to \Omega^{p,q-1}(M)$. We then can define two Laplacians $\Delta^\partial := \partial\partial^* + \partial^*\partial$ and $\Delta^{\overline{\partial}} := \overline{\partial}\overline{\partial}^* + \overline{\partial}^*\overline{\partial}$.

Theorem 22. *If (M,J,h) is Kähler, then $\Delta = 2\Delta^\partial = 2\Delta^{\overline{\partial}}$, where Δ is the usual Laplacian $dd^* + d^*d$.*

E Index Theory

For this section, let (E,π,M) and (F,ϖ,M) be vector bundles over the same space M, unless otherwise stated.

E.1 Differential Operators and the Atiyah-Singer Index Theorem

Definition 27. *Let $D : \Gamma(M,E) \to \Gamma(M,F)$ be a linear operator. We say that D is a* ***differential operator of order*** *k if in local coordinates*

$$Du(x) = \sum_{|\alpha|\le k} D^\alpha(x)\frac{\partial^\alpha x}{\partial x^\alpha} \tag{92}$$

where α is a multi-index, $u \in \Gamma(M,E)$, and $D^\alpha(x)$ a collection of linear maps from $E_x \to F_x$.

Definition 28. *Let $\xi = \{\xi_1, \dots \xi_n\}$ be an element of E_x written in the fiber coordinates of E given by the local trivialization. For a multi-index α with $|\alpha| = k$, we denote by ξ_α the element of $S^k(E_x)$ which is the symmetric product of the ξ_i in accordance with α. The* ***symbol*** *at (x,ξ) of a differential operator D of order k is*

$$\sigma_k(x,\xi) = \sum_{|\alpha|=k} D^\alpha(x)\xi_\alpha \tag{93}$$

Importantly, $\sigma_k(x,\xi)$ is a matrix whose elements are polynomials of $\xi_1,\dots,\xi_n$. If $\sigma_k(x,\xi)$ is invertible for $\xi \neq 0$, we say that D is an ***elliptic differential operator of order*** k.

Given an elliptic operator, there are two very important properties that we would like to focus on. The first is the analytic index.

Definition 29. *The **analytic index** of an elliptic differential operator D is*

$$\operatorname{ind}(D) := \dim(\ker D) - \dim(\ker D^*) \tag{94}$$

where D^* is the formal adjoint of D.

Example 12. *Let $M = \mathbb{R}/\mathbb{Z}$ and $D = \frac{d}{d\theta} - \lambda$ for some complex λ. Then $D^0 = -\lambda$ and $D^1 = 1$, so $\sigma_1(\theta, \xi) = \xi$, implying that D is elliptic. We see that $\ker D = \{0\}$ if λ is not a multiples of $2\pi i$, and $\ker D = \left\{ae^{\lambda\theta} : a \in \mathbb{C}\right\}$. If we look at the adjoint instead, we see that $\ker D^* = \{0\}$ if $\overline{\lambda}$ is not a multiples of $2\pi i$, and $\ker D^* = \left\{ae^{\overline{\lambda}\theta} : a \in \mathbb{C}\right\}$. Since these are the same conditions, we see that $\operatorname{ind}(D) = 0$. So while the dimensions of the kernels of D or D^* may not vary continuously with λ, $\operatorname{ind}(D)$ does. This shows some motivation as to why we would want some kind of operation like* ind.

Consider the disc bundle $B(TM)$ and sphere bundle $S(TM)$ of TM and let $\pi : B(TM) \to M$ be the projection. If $D : E \to F$ is elliptic, we see that $\sigma_k(D)$ defines an isomorphism on $S(TM)$ as follows. Locally, an element of π^*E on $S(TM)$ looks like (ξ, v) for $\xi \in S(TM)$ and $v \in E_\xi$. Then if we think of ξ as a unit vector of appropriate dimension, we can map $\pi^*E \to \pi^*F$ by $(\xi, v) \mapsto (\xi, \sigma(x, \xi)v)$. Since D is elliptic, this map is an isomorphism. With these maps, we can state our second object of interest

Definition 30. *Let M be a compact manifold and TM its tangent space. Local coordinates on TM look like $(x_1, \ldots, x_n, v_1, \ldots, v_n)$. Permute the coordinates to $(x_1, v_1, \ldots, x_n, v_n)$ to define a new orientation on TM*[11]. *Let E, F be complex vector bundles over M, $B(TM)$ be the disc bundle of TM with its new orientation, and $\pi : B(TM) \to M$ be the projection map. The topological index of an elliptic operator $D : E \to F$ is given by*

$$\operatorname{tind}(D) = (-1)^{\dim M} \langle \operatorname{ch}(D), \operatorname{Todd}(TM \otimes \mathbb{C}), [M] \rangle \tag{95}$$

where $\operatorname{ch}(D) := T^{-1}\operatorname{ch}(d(\pi^*E, \pi^*F, \sigma(D)))$.

Proposition 34. *Assuming TM is oriented as above, if $e(TM) \neq 0$ and $\dim M = 2m$, then*

$$\operatorname{tind}(D) = (-1)^m \int_M \frac{\operatorname{ch}(E) - \operatorname{ch}(F)}{e(TM)} \operatorname{Td}(TM \otimes \mathbb{C}) \tag{96}$$

Proof. This is a direct result of Theorem 14. Some simple calculations shows that under our new orientation of TM, $e(TM) \wedge \operatorname{ch}(D) = (-1)^{2m(2m+1)/2}(\operatorname{ch}(E) - \operatorname{ch}(F))$. Since $e(TM) \neq 0$, we can make sense of the division above. The desired result then follows. $\square$

A crowning achievement of 20th century differential geometry is the Atiyah-Singer index theorem.

Theorem 23. *For any elliptic operator $D : E \to F$ between complex vector bundles over a compact manifold M,* $\operatorname{ind}(D) = \operatorname{tind}(D)$.

E.2 Consequences of the Atiyah-Singer Index Theorem

We will first provide an alternative proof of the Chern-Gauss-Bonnet Theorem using index theory. We assume that M is an even-dimensional Riemannian manifold.

Let $D = d + d^* : \Lambda^{\text{even}}(T^*M) \to \Lambda^{\text{odd}}(T^*M)$, as defined in Section 1.3. We claim that D_{even} is an elliptic differential operator.

Proposition 35. *The index of D_{even} is the Euler characteristic of M.*

[11] The reason for this change is so that the orientation on TM is compatible with a certain complex structure.

Proof. This is exactly Corollary 3. ◻

Proposition 36. *If $D = d + d^*$, then*

$$\operatorname{tind}(D_{even}) = \int_M e(TM) \tag{97}$$

Proof. We will assume that the genus of M is not zero. Set $E = \Lambda^{\text{even}}(T^*M)$ and $F = \Lambda^{\text{odd}}(T^*M)$. By Theorem 5, we know that $TM \otimes \mathbb{C}$ decomposes as $L_1 \oplus \overline{L_1} \oplus \cdots \oplus L_m \oplus \overline{L_m}$ for some complex line bundles $L_1, \ldots, L_m$. Define the chern roots $x_1(TM \otimes \mathbb{C}), \ldots, x_{2m}(TM \otimes \mathbb{C})$ as $x_1(TM \otimes \mathbb{C}), \ldots, x_m(TM \otimes \mathbb{C}) = c_1(L_1), \ldots, c_1(L_m)$ and $x_{m+1}(TM \otimes \mathbb{C}), \ldots, x_{2m}(TM \otimes \mathbb{C}) = c_1\left(\overline{L_1}\right), \ldots, c_1\left(\overline{L_n}\right)$ The latter collection are equal to$-c_1(L_1), \ldots, -c_1(L_m)$. We see then that

$$\begin{aligned}
\operatorname{ch}(E) - \operatorname{ch}(F) &= 1 - \operatorname{ch}(T^*M \otimes \mathbb{C}) + \operatorname{ch}\left(\Lambda^2 T^*M \otimes \mathbb{C}\right) + \ldots + (-1)^{2m} \operatorname{ch}\left(\Lambda^{2m} T^*M \otimes \mathbb{C}\right) \\
&= 1 - \sum_i e^{-x_i} + \sum_{i<j} e^{-x_i} e^{-x_j} + \cdots + (-1)^{2m} e^{-x_1} \cdots e^{-x_{2m}} (TM \otimes \mathbb{C}) \\
&= \prod_{i=1}^{2m} \left(1 - e^{-x_i}\right)(TM \otimes \mathbb{C})
\end{aligned}$$

with the second line being true by using the duality of the chern roots. Similarly,

$$\begin{aligned}
\operatorname{Td}(TM \otimes \mathbb{C}) &= \operatorname{Td}(L_1) \cdots \operatorname{Td}(L_m) \operatorname{Td}\left(\overline{L_1}\right) \cdots \operatorname{Td}\left(\overline{L_m}\right) \\
&= \frac{x_1}{1 - e^{-x_1}} \cdots \frac{x_m}{1 - e^{-x_m}} \frac{x_{m+1}}{1 - e^{-x_{m+1}}} \cdots \frac{x_{2m}}{1 - e^{-x_{2m}}} (TM \otimes \mathbb{C}) \\
&= \prod_{i=1}^{2m} \frac{x_i}{1 - e^{-x_i}} (TM \otimes \mathbb{C})
\end{aligned}$$

We also know that $e(TM) = c_m(TM \otimes \mathbb{C}) = \prod_{i=1}^m x_i (TM \otimes \mathbb{C})$

$$\begin{aligned}
\operatorname{tind}(D_{\text{even}}) &= (-1)^m \int_M \frac{\operatorname{ch}(E) - \operatorname{ch}(F)}{e(TM)} \operatorname{Td}(TM) \\
&= (-1)^m \int_M \frac{\prod_{i=1}^{2m} \left(1 - e^{-x_i}\right)}{\prod_{i=1}^m x_i} \prod_{i=1}^{2m} \frac{x_i}{1 - e^{-x_i}} (TM \otimes \mathbb{C}) \\
&= (-1)^m \int_M \prod_{i=m+1}^{2m} x_i (TM \otimes \mathbb{C}) \\
&= (-1)^{2m} \int_M \prod_{i=1}^{m} x_i (TM \otimes \mathbb{C}) \\
&= \int_M e(TM)
\end{aligned}$$

◻

Corollary 16 (The Chern-Gauss-Bonnet Theorem)**.** *The characteristic number of* $e(TM)$ *is the Euler characteristic.*

Proof. By Theorem 23, ◻

$$\begin{aligned}
\langle e(TM), [M] \rangle &= \int_M e(TM) \\
&= \operatorname{tind}(D_{\text{even}}) \\
&= \operatorname{ind}(D_{\text{even}}) \\
&= \chi(M)
\end{aligned}$$

In a similar vein, let us prove the following theorem.

Theorem 24 (Hirzebruch–Riemann–Roch theorem)**.** *If M is a complex manifold of dimension n and V is a holomorphic vector bundle over M, then*

$$\chi(M,V)=\int_M \mathrm{ch}(V)\,\mathrm{Td}(TM) \tag{98}$$

Proof. The proof of the Hirzebruch–Riemann–Roch theorem is very similar to the proof of the Chern-Gauss-Bonnet theorem (which we might expect since we want to equate the Euler characteristic to a characteristic number). Let $D=\overline{\partial}+\overline{\partial}^*$, where $\overline{\partial}$ comes from the holomorphic vector bundle. Define D_{even} to the restriction of D to maps from even forms to odd forms. A proof identical to Corollary 3 shows that $\mathrm{ind}(D_{even})=\chi(X,V)$.

The harder part is computing the topological index. Let $x_i(TM)$ be the chern roots of TM. We see that $e(TM)=\prod_{i=1}^n x_i(TM)$ as before. Further, if we abbreviate $\Lambda^{0,\text{even}}(T^*M)$ to $\Lambda^{0,\text{even}}$ and $\Lambda^{0,\text{odd}}(T^*M)$ to $\Lambda^{0,\text{odd}}$, then

$$\begin{aligned}\mathrm{ch}\left(V\otimes\Lambda^{0,\text{even}}\right)-\mathrm{ch}\left(V\otimes\Lambda^{0,\text{odd}}\right)&=\mathrm{ch}(V)\,\mathrm{ch}\left(\Lambda^{0,\text{even}}\right)-\mathrm{ch}(V)\,\mathrm{ch}\left(\Lambda^{0,\text{odd}}\right)\\&=\mathrm{ch}(V)\prod_{i=1}^n\left(1-e^{x_i}\right)(TM)\end{aligned}$$

Finally,

$$\begin{aligned}\mathrm{Td}(TM\otimes\mathbb{C})&=\mathrm{Td}\left(T^{1,0}M\oplus T^{0,1}M\right)\\&=\mathrm{Td}\left(T^{1,0}M\right)\mathrm{Td}\left(T^{0,1}M\right)\\&=\prod_{i=1}^n\frac{x_i}{1-e^{-x_i}}(TM)\prod_{j=1}^n\frac{-x_j}{1-e^{x_j}}(TM)\end{aligned}$$

Putting it all together, we find

$$\begin{aligned}\chi(X,V)&=(-1)^n\int_X\frac{\mathrm{ch}\left(V\otimes\Lambda^{0,\text{even}}(T^*M)\right)-\mathrm{ch}\left(V\otimes\Lambda^{0,\text{odd}}(T^*M)\right)}{e(TM)}\mathrm{Td}(TM\otimes\mathbb{C})\\&=(-1)^n\int_X\frac{\mathrm{ch}(V)\prod_{i=1}^n\left(1-e^{x_i}\right)(TM)}{\prod_{i=1}^n x_i(TM)}\prod_{i=1}^n\frac{x_i}{1-e^{-x_i}}(TM)\prod_{j=1}^n\frac{-x_j}{1-e^{x_j}}(TM)\\&=(-1)^n\int_X\mathrm{ch}(V)\prod_{i=1}^n\frac{1}{1-e^{-x_i}}(TM)\prod_{j=1}^n(-x_j)(TM)\\&=(-1)^{2n}\int_X\mathrm{ch}(V)\prod_{i=1}^n\frac{x_i}{1-e^{-x_i}}(TM)\\&=\int_X\mathrm{ch}(V)\,\mathrm{Td}(TM)\end{aligned}$$

□

Corollary 17 (Riemann-Roch)**.** *If X is a Riemann surface and L is an analytic line bundle over X, then $\chi(X,L)=c_1(L)+1-g$.*

Proof. While we have already seen a proof of this theorem, we will now present one using the Hirzebruch–Riemann–Roch theorem. Since X is a Riemann surface, $\mathrm{td}(TX)=1+\frac{c_1(TX)}{2}$.

Similarly, since L is a line bundle, we have that $\mathrm{ch}(L) = 1 + c_1(L)$. As such

$$\begin{aligned}\chi(X, L) &= \int_X \mathrm{ch}(L)\,\mathrm{td}(TX) \\ &= \int_X (1 + c_1(L))\left(1 + \frac{c_1(TX)}{2}\right) \\ &= \int_X c_1(L) + \frac{c_1(TX)}{2} \\ &= \int_X c_1(L) + \frac{e(TX)}{2} \\ &= c_1(L) + 1 - g\end{aligned}$$

$\square$

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